

The Mathematical Principles of Natural Philosophy



Isaac Newton · 1687

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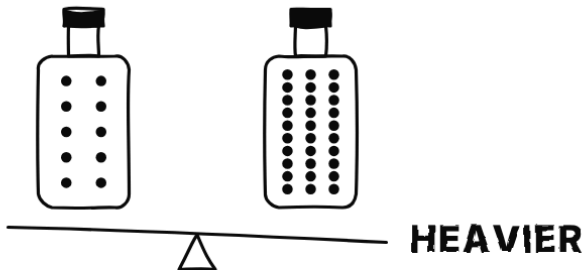
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How Much Matter, How Much Motion

Take two identical stiff plastic bottles, one holding air at ordinary pressure with its cap screwed on. Now pump air into the other with a bicycle pump until it is hard. Both bottles take up the same space, yet the hard one holds more air, and on a sensitive kitchen scale it even weighs a little more. So "how much" of something there is cannot be told by its size alone. That is the question Newton settles first, before a single law of motion is written down.



$$\text{MASS } m = \underset{\text{DENSITY}}{\rho} \times V \text{ VOLUME}$$

$$2 \times 2 = 4$$

$$2 \times 3 = 6$$

Same Size, More Matter

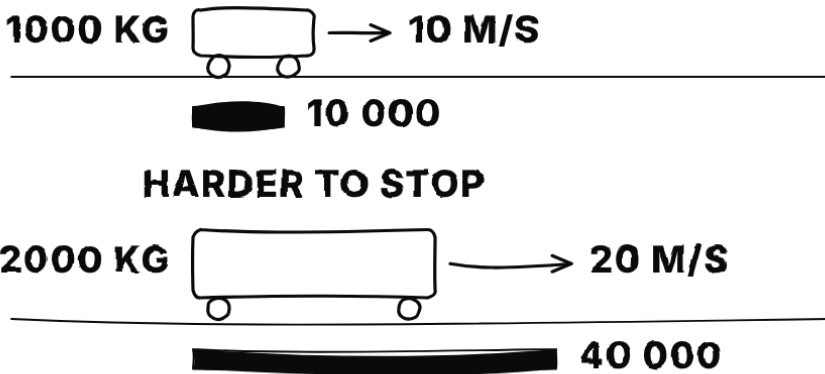
His first definition says that the quantity of matter is measured by its density and its bulk taken together. In the notation of today one writes it as $m = \rho V$: the mass m is the density ρ , the Greek letter rho, multiplied by the volume V . In words, it says that a body has more matter when it is packed more tightly, and more when it is bigger, and that the two effects multiply. The intuition is a crowd: twice as many people per square metre on twice as many square metres gives four times as many people. Newton's own numbers work the same way: air of double density in a double space is four times the quantity, because $2 \times 2 = 4$; in a triple space it is six times, because $2 \times 3 = 6$. Snow pressed into ice and powder squeezed into a cake change their bulk and their density, but not their quantity of matter.

Newton sets aside, for the moment, any thin medium that might flow freely between the parts of bodies; he counts only the matter of the body itself. This quantity is what he will call, from now on, body or mass. And he adds something that is not obvious at all: the mass of a body is proportional to its weight. He found that by experiments on pendulums, made very carefully and described later in the book. Pendulums of the same length, with bobs of very different materials, swing in the same time, and that is what tells him that weight grows exactly in step with mass. It is the reason the scale in your bathroom, which really measures a pull, can be printed in kilograms of matter.

The second definition builds on the first. The quantity of motion is measured by the velocity and the quantity of matter together, and today we write $p = mv$, where p is called momentum. In words: the more matter is moving, and the faster it moves, the more motion there is. The intuition is that the motion of a whole body is just the motion of all its parts added up, so twice the matter at the same speed is twice the motion. Newton's numbers again: a body of double

quantity at equal velocity has double motion; with twice the velocity as well, it has four times the motion, $2 \times 2 = 4$. A car of 1000 kilograms at 10 metres per second carries $1000 \times 10 = 10\,000$ units of motion; a van of 2000 kilograms at 20 metres per second carries $2000 \times 20 = 40\,000$, four times as much, which is why it needs so much more road to stop.

$$\begin{array}{ccc}
 p & = & m \quad v \\
 \text{MOMENTUM} & & \text{MASS} \quad \text{SPEED}
 \end{array}$$



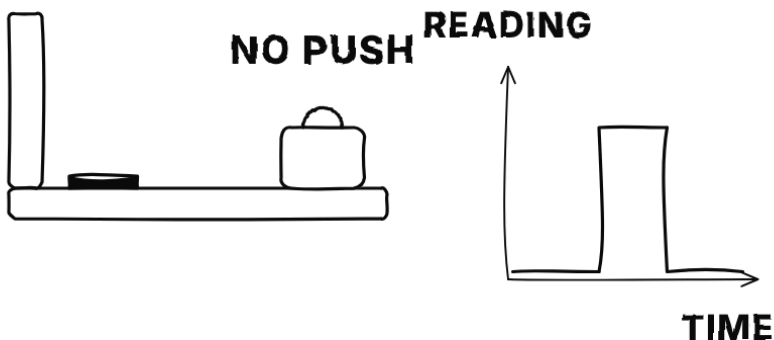
Motion Is Mass Times Speed

The third definition gives a name to something everyone feels but nobody had defined: the vis insita, the force that is born in matter itself. It is a power of resisting, by which every body, as far as it can, keeps its present state, whether at rest or moving uniformly in a straight line. Newton insists that it is proportional to the body, and that it is nothing but the inactivity of the mass, seen from another

side. His better name for it is *vis inertiae*, the force of inactivity, and from it comes our word inertia. A body shows this force only when some other force tries to change its state. Seen from the body that holds still, it looks like resistance; seen from the body that pushes into another, it looks like impulse. And Newton warns that rest and motion, as people usually think of them, are distinguished only relatively; bodies that look at rest are not always truly at rest.

Try it yourself

Put your phone flat on the passenger seat of a car and open any free sensor app that shows the accelerometer. While the car rolls steadily along a straight road, the reading in the direction of travel stays close to zero. When the driver brakes, the reading jumps, and a loose bag on the seat slides forward. Nothing pushed the bag forward; it simply kept going while the car was made to stop, which is the *vis inertiae* at work.

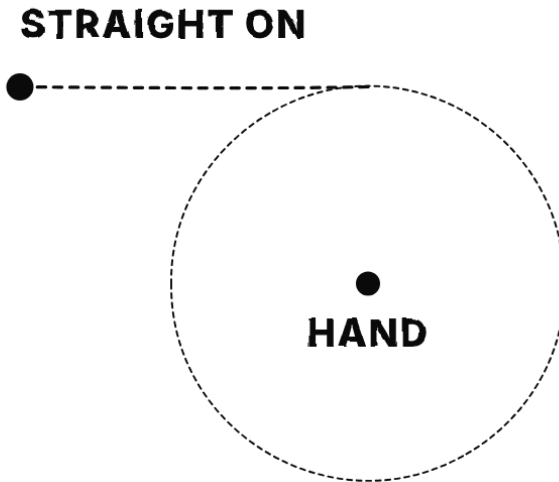


The Bag Keeps Going

The fourth definition: an impressed force is an action on a body that changes its state of rest or of uniform straight motion. It exists only while the action lasts, and nothing of it stays in the body afterwards; the body keeps its new state by its inertia alone. Impressed forces come from blows, from pressure, and from centripetal force. The brake pads press only while the pedal is down; once the car has slowed to its new speed, it keeps that speed with nothing pressing at all.

The fifth definition introduces the force the whole book is about: a centripetal force draws or impels a body towards a point, as towards a centre. Gravity is one, pulling bodies towards the centre of the earth; magnetism is another, pulling iron towards the lodestone; and so is

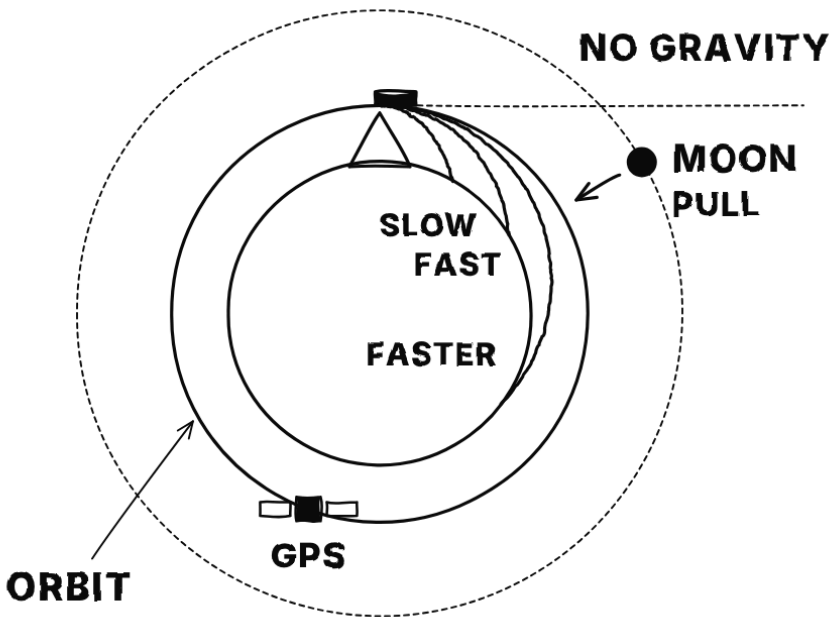
the force, whatever it may be, that keeps bending the planets out of straight lines into curved orbits. Newton's picture is a stone whirled in a sling. The stone strains to fly away from the hand and stretches the sling, the harder the faster it goes, and the moment it is released it flies off. The pull of the sling, always directed back towards the hand at the centre of the circle, is what he calls centripetal force, literally the force that seeks the centre. Every body moving in any orbit would leave it along a straight line, uniformly, if such a force did not hold it in.



The Pull Toward the Centre

Then comes the most famous picture of the definitions: a cannon on top of a mountain, firing a lead ball horizontally. Without gravity, and without air, the ball would go straight on forever; gravity bends its

path downwards, the more the heavier it is for its mass, the less the faster it is fired. Newton gives numbers: if the ball lands two miles away, then with no air, twice the speed carries it about four miles and ten times the speed about twenty. The rule behind that is simple: the ball falls the same height in the same time whatever its speed, so the distance it covers before landing grows in proportion to the speed. Keep increasing the speed and the path flattens, until the ball lands 10, 30 or 90 degrees around the earth, goes all the way round before falling, or never falls at all and travels on into the heavens. That last case is not a fantasy: every GPS satellite your phone listens to is Newton's cannonball, moving so fast sideways that it keeps falling around the earth without ever coming down.



Newton's Cannon

The moon is held the same way. If the force pulling it towards the earth were too weak, the moon would drift off along a straight line; if it were too strong, it would be pulled down out of its orbit. The force has to be of exactly the right size for the speed the moon has. And here Newton states the programme of his whole book: it belongs to the mathematicians to find the force that keeps a body in a given orbit at a given speed, and, the other way round, to find the curved path a body takes when it is thrown from a given place, at a given speed, under a given force. Engineers who place a satellite in orbit solve exactly these two problems, the force for a path and the path for a force, before every launch.

Remember

Newton measures matter by density and volume together, so double density in double space is four times the matter, and this quantity, proportional to weight, is what he calls mass. The quantity of motion is mass and velocity together, so twice the mass at twice the speed is four times the motion. Inertia, the *vis inertiae*, is a body's tendency to keep its state, and an impressed force exists only while it acts. A centripetal force pulls towards a centre and turns straight motion into an orbit, as the sling holds the stone and gravity holds the moon. Fire a cannonball fast enough and it never lands, which is how every satellite stays up, and finding forces from orbits and orbits from forces is the task the book sets itself.

Questions

1. Using $m = \rho V$, what happens to the quantity of matter if the density is doubled and the volume is tripled?
 - a. It doubles
 - b. It becomes five times as large
 - c. It becomes six times as large
 - d. It stays the same, since mass cannot change
2. Why could Newton conclude from pendulums that mass is proportional to weight?
 - a. Heavier bobs swing faster, in step with their weight
 - b. Pendulums of the same length swing in the same time whatever their bobs are made of
 - c. Pendulums stop sooner when their bobs are light
 - d. A pendulum's swing measures its density directly
3. A loose bag slides forward when the car brakes. What does the lecture say is happening?
 - a. The bag keeps going by its vis inertiae while the car is made to stop
 - b. The brakes push the bag forward through the seat
 - c. The accelerometer's reading pushes the bag
 - d. A centripetal force pulls the bag towards the windscreen
4. If air were removed, why does the cannonball's range grow in proportion to its speed?
 - a. A faster ball feels a weaker gravity

- b. A faster ball has more inertia, so it falls more slowly
- c. The ball falls the same height in the same time whatever its speed, so the distance grows with the speed
- d. Gravity only acts once the ball has stopped moving forward

Answers

- 1. **c)** Density and bulk multiply, so twice the density in three times the space gives two times three, six times the matter, as Newton says of air in a triple space.
- 2. **b)** Pendulums of the same length with bobs of very different materials swing in the same time, which tells him that weight grows exactly in step with mass.
- 3. **a)** Nothing pushed the bag forward: it simply kept going while the car was made to stop, which is the vis inertiae at work.
- 4. **c)** The ball falls the same height in the same time whatever its speed, so the distance it covers before landing grows in proportion to the speed.

Worked examples

Air of double density Air of double density fills a double space, and in a second case a triple space. How many times the original quantity of matter is each, using $m = \rho V$?

In $m = \rho V$ the density and the volume multiply.

Double density, double volume: $2 \times 2 = 4$, four times the matter.

Double density, triple volume: $2 \times 3 = 6$, six times the matter.

Answer: four times and six times

Newton's cannonball A lead ball fired level from a mountain lands two miles away. Without air, how far does it go at twice and at ten

times the speed?

Without air, the ball falls the same height in the same time at any speed.

So the distance it covers before landing is proportional to the speed.

Twice the speed: $2 \times 2 = 4$ miles; ten times the speed: $10 \times 2 = 20$ miles.

Answer: 4 miles and 20 miles

Practice

1. Using $p = mv$, a body of double mass moves at twice the speed of another. How many times its quantity of motion does it carry?

Hint: Multiply the factor for the mass by the factor for the speed.

Answer: Four times: $2 \times 2 = 4$, as Newton says, quadruple.

2. A car of 1000 kilograms moves at 10 metres per second and a van of 2000 kilograms at 20 metres per second. How many times the car's momentum does the van have?

Hint: Work out $p = mv$ for each and divide.

Answer: Four times: the car has $1000 \times 10 = 10\,000$ units, the van $2000 \times 20 = 40\,000$ units.

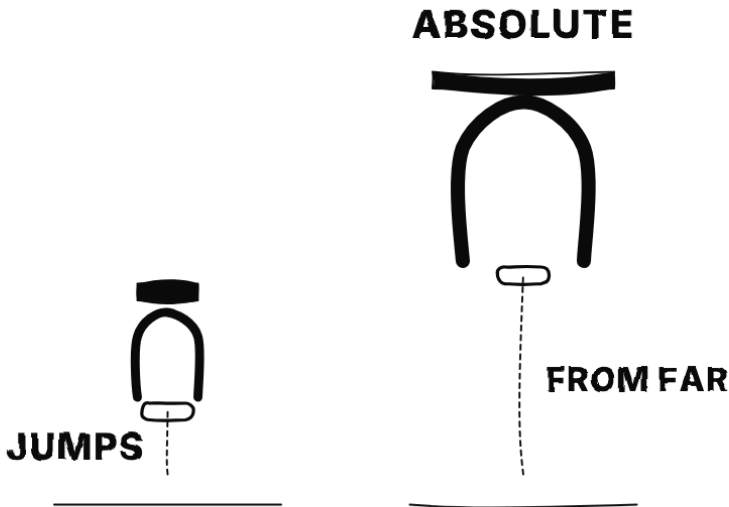
3. Without air, the ball that lands two miles away is fired at three times the speed. How far does it go?

Hint: Range grows in proportion to speed when the fall takes the same time.

Answer: Six miles: three times the speed carries it three times as far, $3 \times 2 = 6$ miles.

Three Ways to Measure One Pull

Hold a small fridge magnet a few centimetres above a paper clip and lower it slowly. At some height the clip jumps up and sticks. Try again with a big, strong magnet and the clip jumps from much farther away. And with either magnet, the closer you hold it, the more sharply the clip leaps. So "how strong is the pull?" has more than one honest answer, and Newton decides to keep three of them apart.



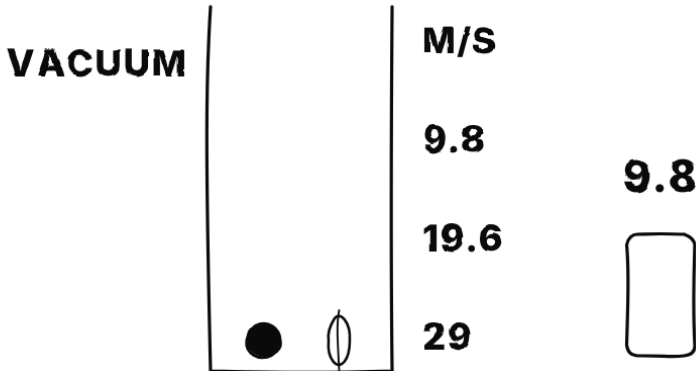
Small Magnet, Big Magnet

The first is the absolute quantity of a centripetal force: a measure of the source itself, of how powerful the cause is that sends the pull out

from the centre into the space around it. In words, a big strong loadstone has more absolute force than a small weak one, whatever you put near it. There is no single formula in the passage for it; it simply ranks the sources against each other. Your phone's compass feels the earth's magnetism as a whole, a source far stronger than any fridge magnet, though it is spread so wide that the needle barely notices.

The second is the accelerative quantity: a measure proportional to the velocity the pull produces in a given time. Today we would write it as $a = \Delta v / \Delta t$, the acceleration, where Δ , the Greek letter delta, means "change in". It says how quickly the pull speeds things up at a given place, whatever is placed there. The same loadstone accelerates more strongly close up than far away, and gravity is a little stronger in a valley than on top of a very high mountain, and weaker still farther from the earth. But at the same distance from the earth, once the air is allowed for, gravity speeds up every falling body equally, heavy or light, big or small. A calculation: a stone dropped near the ground gains about 9.8 metres per second of speed each second, so after 3 seconds it falls at about $9.8 \times 3 \approx 29$ metres per second, and a feather in a vacuum tube would do exactly the same. The accelerometer in your phone, lying still on a table, reads that same 9.8 at every place at the same height, whether the phone is bare or inside a heavy case.

$$\text{ACCELERATION } a = \frac{\Delta v \text{ SPEED GAIN}}{\Delta t \text{ TIME}}$$



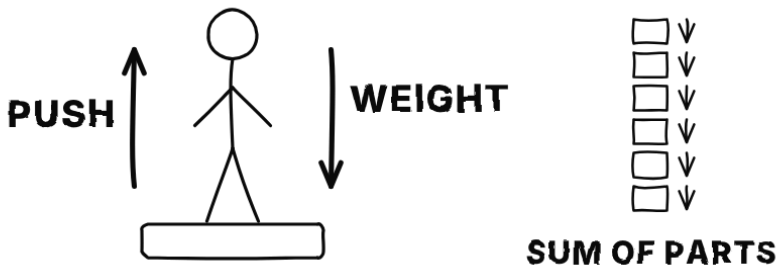
Acceleration: Speed Gained per Second

The third is the motive quantity: a measure proportional to the motion, the momentum, that the pull produces in a given time. This is what Newton calls the body's weight, its whole tendency towards the centre, and it is always measured by the equal and opposite force that is just enough to stop the body from falling. The bathroom scale does exactly that: it pushes up just hard enough to stop you from sinking, and reads out how hard that is. A bigger body has more weight, and the same body weighs more near the earth and less far from it.

$$F = m a$$

$$W = m g$$

M: MASS, G: 9.8



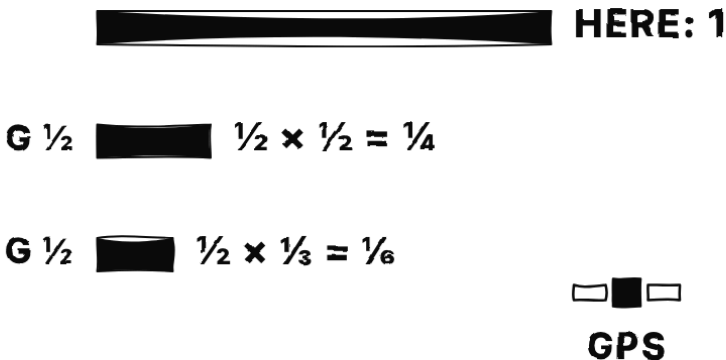
Weight Is Mass Times g

Newton ties the three together. The motive force belongs to the body, as the sum of the tendencies of all its parts; the accelerative force belongs to the place, as a kind of power spread from the centre to every spot around it; and the absolute force belongs to the centre, as whatever cause produces the pull. That cause might be a central body, a loadstone or the earth, or something not yet known; he says plainly that he gives only a mathematical notion of these forces, not their physical causes.

Then comes the key relation: the accelerative force is to the motive force as speed is to motion. Just as motion is speed multiplied by the quantity of matter, so the motive force is the accelerative force multiplied by the quantity of matter, which today is written $F = ma$,

and for weight $W = mg$. In words: the pull on a whole body is the pull on each little part, added up over all its parts. That is why near the earth, where the accelerative gravity g is the same for everything, weight is simply proportional to the body.

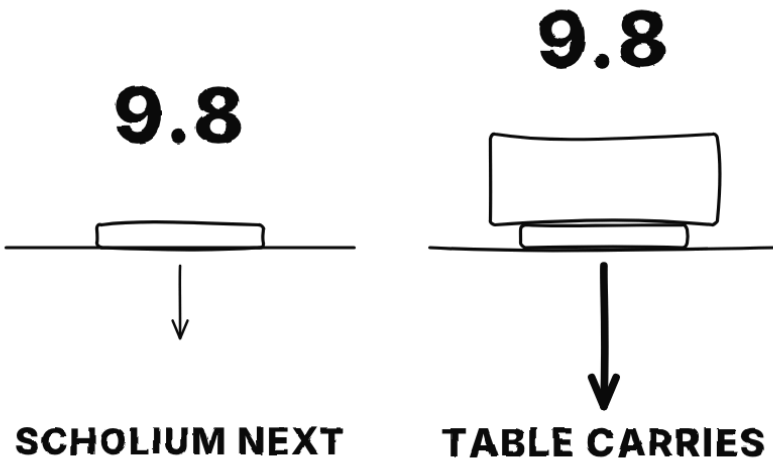
Newton's own calculation goes higher up. Suppose we rise to a region where the accelerative gravity is only half of what it is here. A body two times smaller there weighs $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the big body's weight here, four times less; a body three times smaller weighs $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, six times less. At the height of the GPS satellites, about twenty thousand kilometres up, the accelerative gravity is far below the 9.8 we feel, so a satellite there weighs only a small fraction of what it weighed on the launch pad, though its mass is exactly the same.



Half the g , Smaller Bodies

Try it yourself

Put your phone flat on a table and open a free sensor app that shows the accelerometer. Note the reading pointing up from the table, close to 9.8 metres per second each second. Now lay a heavy book on the phone and look again: the reading does not change. The accelerative quantity belongs to the place, not to the body; the book's weight changed what the table carries, not the acceleration gravity gives there.



The Book Does Not Change g

Newton adds a warning about words: he will call attractions and impulses accelerative and motive in the same sense, and use

attraction, impulse and propensity towards a centre interchangeably. He does not claim to say what these forces are or how they act, nor that a centre, which is only a mathematical point, truly attracts anything. And here the definitions end and a long note, a scholium, begins. Time, space, place and motion, he says, need no definition, being known to all; but ordinary people think of them only through the things they can see and touch, and that breeds prejudices. To remove them he will separate absolute from relative, true from apparent, mathematical from common.

Remember

A pull towards a centre can be measured three ways: by the power of its source, the absolute quantity; by how fast it speeds things up at a place, the accelerative quantity; and by the motion it produces in a whole body, the motive quantity, which is weight. Motive force is accelerative force times quantity of matter, as motion is speed times quantity of matter, so where gravity's acceleration is equal, weight is simply proportional to the body. Where gravity is halved, a body two or three times smaller weighs four or six times less. At the same distance from the earth gravity accelerates every body alike, which a phone's accelerometer shows whether or not a book lies on it. Newton gives these forces a mathematical meaning only, and turns next to separating true time, space and motion from their everyday measures.

Questions

1. A phone's accelerometer reads the same 9.8 with or without a heavy book on it. Which of Newton's three quantities does that reading show?
 - a. The motive quantity, which grows with the body
 - b. The accelerative quantity, which belongs to the place
 - c. The absolute quantity, which belongs to the source
 - d. The quantity of motion of the book
2. Using $F = ma$, why is weight simply proportional to the body near the surface of the earth?
 - a. Because every body there has the same mass
 - b. Because the accelerative gravity g is the same for everything there
 - c. Because weight does not depend on gravity
 - d. Because the motive force is the same for all bodies
3. Which quantity measures how powerful the source of a pull is, whatever is placed near it?
 - a. The motive quantity
 - b. The accelerative quantity
 - c. The absolute quantity
 - d. The quantity of motion
4. In a region where the accelerative gravity is halved, how does the weight of a body three times smaller compare with the larger body's weight here?

- a. Three times less
- b. Five times less
- c. Six times less
- d. Nine times less

Answers

- 1. **b)** The accelerative quantity belongs to the place, not to the body; the book changed what the table carries, not the acceleration gravity gives there.
- 2. **b)** Near the earth the accelerative gravity g is the same for everything, so the weight $W = mg$ grows in proportion to the quantity of matter alone.
- 3. **c)** The absolute quantity is a measure of the source itself, of how powerful the cause is that sends the pull out from the centre.
- 4. **c)** Weight is mass times accelerative gravity, so a third of the mass times half the gravity gives a sixth, six times less.

Worked examples

Weight where gravity is halved Where the accelerative gravity is half of its value here, how much do bodies two and three times smaller weigh, compared with the larger body here?

Weight is $W = mg$: the quantity of matter times the accelerative gravity.

Two times smaller and half the gravity: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, four times less.

Three times smaller and half the gravity: $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$, six times less.

Answer: four times less and six times less

Practice

- 1. A stone dropped near the ground gains about 9.8 metres per second of speed every second. How fast does it fall after 2 seconds, ignoring

the air?

Hint: The accelerative quantity is the speed gained per second; multiply by the time.

Answer: About 19.6 metres per second: $9.8 \times 2 = 19.6$.

2. Where the accelerative gravity is halved, a body is four times smaller than one here. How many times less does it weigh?

Hint: Use $W = mg$ and multiply the two fractions.

Answer: Eight times less: $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.

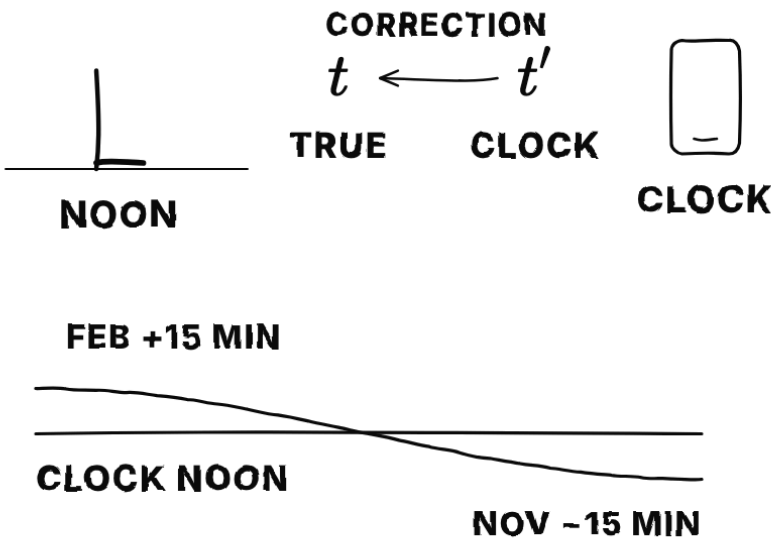
3. Near the earth, one body has three times the quantity of matter of another. How do their weights compare?

Hint: Near the earth g is the same for both, so $W = mg$ depends only on m .

Answer: Three times the weight: with the same g , weight is proportional to the quantity of matter.

True Time, True Space, and the Sailor

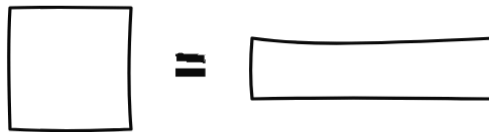
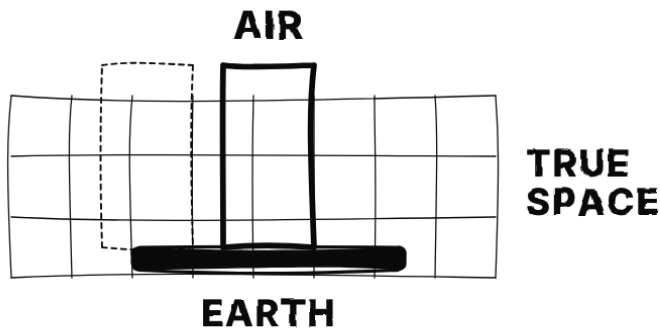
Put a sundial, or just a stick in the ground, in your garden, and note the moment its shadow is shortest each day: local noon. Compare it with your phone's clock across a year. Against the steady clock, the shadow's noon drifts through the year: in February it comes about a quarter of an hour later than its average time, in early November about a quarter of an hour earlier. So which one is the real noon, and what is a day, if days by the sun are not all the same length?



Noon by the Shadow, Noon by the Clock

Newton's scholium answers by splitting time in two. Absolute, true, mathematical time flows evenly of itself, without regard to anything outside, and its other name is duration. Relative, common time is any visible measure of duration made by some motion, an hour, a day, a month, a year, accurate or not. In symbols one might write true time as t and a measured time as t' , read "t prime"; the clock gives t' , and only after correction does it tell t . Your phone's clock is a t' : it counts the vibrations of a quartz crystal and is corrected against the satellites' atomic clocks, but it is still a motion used to measure time.

Then space. Absolute space, of its own nature, stays always the same and never moves. Relative space is a movable measure of it, fixed by the senses through its position among bodies, such as the space under the ground, the air, or the sky, measured from the earth. The two have the same shape and size, but they need not stay the same stretch of space: if the earth moves, the air that keeps the same position around us passes, absolutely speaking, through one part of space after another.



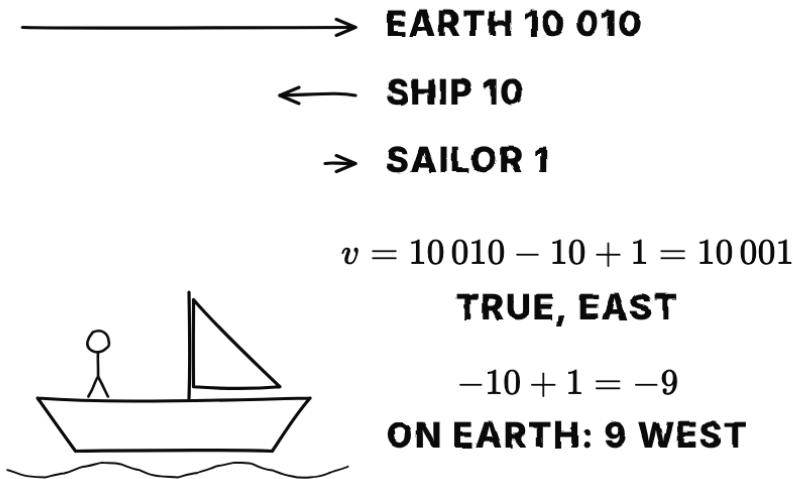
SAME PLACE

True Space and Places

Place is the part of space a body fills, absolute or relative as the space is. It is a part of space, not the body's position nor its outer surface: equal solids always have equal places, though their surfaces, being differently shaped, are often unequal. Positions have no size; they are properties of places rather than places. And since the motion of a whole is the sum of the motions of its parts, the place of a whole is the sum of the places of its parts, inside the whole body.

Motion follows. Absolute motion is the passage of a body from one absolute place to another; relative motion is the passage from one relative place to another. On a ship under sail, the relative place of a body is the part of the ship it fills, which moves with the ship, and relative rest is staying in that part. True rest would be staying in the

same part of immovable space, through which the ship and everything in it is carried. So if the earth is truly at rest, a body resting in the ship really moves with the ship's speed over the earth; and if the earth moves too, the body's true motion is made up of the earth's motion, the ship's motion over the earth, and the body's own motion in the ship.



The Sailor's Velocity

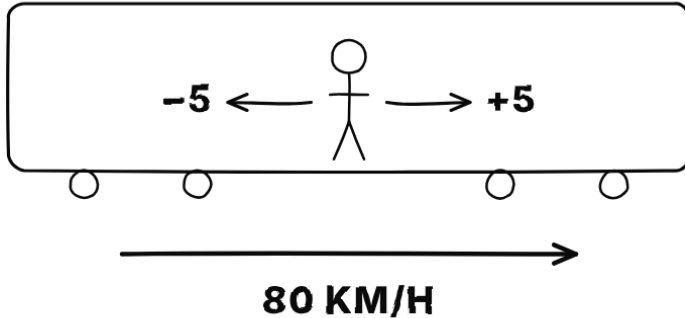
Newton puts numbers on it. Let the part of the earth under the ship move truly eastwards with 10 010 parts of velocity, the ship sail westwards with 10 parts, and a sailor walk eastwards in the ship with 1 part. Counting east as positive, the sailor's true velocity is $v = 10\ 010 - 10 + 1 = 10\ 001$ parts eastwards. In words: velocities along the same line simply add, each one measured from the thing that carries it. His

velocity relative to the earth leaves out the earth's own share: $-10 + 1 = -9$, nine parts westwards. So one and the same sailor is rushing east through space while drifting west along the shore. A phone's GPS shows exactly the second kind of number: your speed over the ground, which leaves out the earth's own motion entirely.

Try it yourself

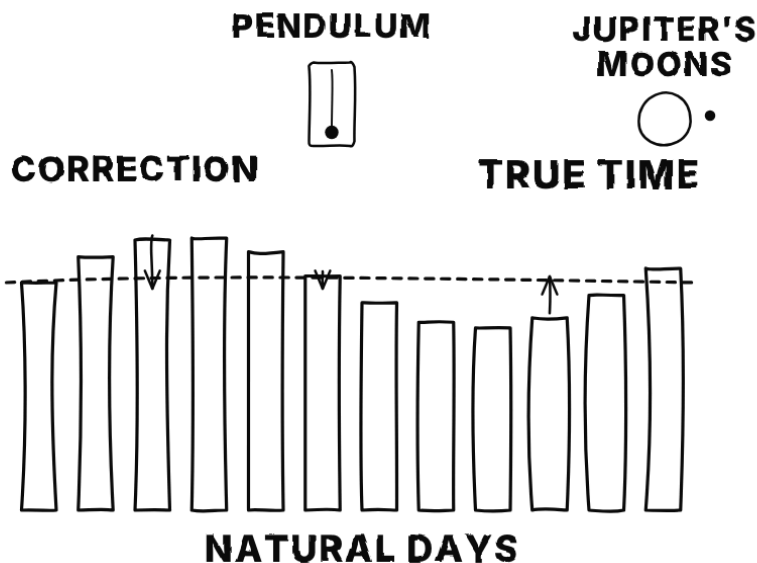
Walk forward along the aisle of a moving train with a GPS speed app open. The app shows the train's speed plus your walking pace, perhaps 80 kilometres per hour plus 5, so 85. Walk towards the back and it shows 80 minus 5, so 75. The app is adding velocities along a line, just as Newton does with the sailor, and it stops at the ground, never including the earth's own motion.

BACK: 75 FORWARD: 85



Walking in a Train

Back to time, where astronomy draws the line in practice. Absolute time is separated from relative time by the equation, or correction, of common time. The natural days, from sun to sun, are truly unequal, though people treat them as equal and use them to measure time; astronomers correct this inequality to compute the motions of the heavens accurately. That correction is what your sundial showed: about a quarter of an hour either way at different seasons.



Natural Days Are Unequal

It may be, Newton adds, that no motion is perfectly even, so that time can never be measured exactly; every motion may speed up or slow down, but the flow of true time cannot change. The duration of things is the same whether their motions are fast, slow, or absent altogether, so duration must be kept apart from its visible measures, and recovered from them by the astronomers' correction. Two things prove that the correction is needed: experiments with pendulum clocks, and the eclipses of the moons of Jupiter, whose timing only fits once the unequal days are corrected. The satellite clocks that keep your phone's time are corrected in the same spirit, so that what they count matches the steady time that the calculations need.

Remember

True time flows evenly by itself, and every clock, a day, a pendulum or your phone, is only a motion used to measure it. True space never moves; the spaces we measure are fixed by bodies and may be travelling through it. A place is the part of space a body fills, and true motion is passage between places of true space, while relative motion is measured against moving surroundings. Newton's sailor goes 10 010 minus 10 plus 1, that is 10 001 parts east truly, while going 9 parts west relative to the earth. Astronomers recover true time from unequal natural days by a correction, which pendulum clocks and the eclipses of Jupiter's moons show to be necessary.

Questions

1. Your phone's clock is called a t' rather than t . Why?
 - a. Because it is always wrong
 - b. Because it is a measure of duration made by some motion, not true time itself
 - c. Because it measures space, not time
 - d. Because it only counts days
2. On a moving ship, what is a body's relative rest?
 - a. Staying in the same part of immovable space
 - b. Staying in the same part of the ship, which moves with the ship

- c. Moving with the ship's velocity over the earth
 - d. Moving against the wind
3. The earth moves east with 10 010 parts, the ship west with 10, the sailor east in the ship with 1. What is the sailor's velocity relative to the earth?
- a. 10 001 parts east
 - b. 11 parts west
 - c. 9 parts west
 - d. 1 part east
4. What do pendulum clocks and the eclipses of Jupiter's moons show, according to Newton?
- a. That absolute time does not exist
 - b. That the correction of unequal natural days is needed
 - c. That the earth is at rest
 - d. That all motions are perfectly even

Answers

1. **b)** Relative time is any visible measure of duration made by some motion; the phone counts a crystal's vibrations, so it gives t' , which only after correction tells t .
2. **b)** The relative place of a body is the part of the ship it fills, which moves with the ship, and relative rest is staying in that part.
3. **c)** Relative to the earth, the earth's own share drops out: $-10 + 1 = -9$, nine parts westwards.
4. **b)** Two things prove that the correction is needed: experiments with pendulum clocks and the eclipses of the moons of Jupiter.

Worked examples

The sailor's two velocities The part of the earth under the ship moves east with 10 010 parts of velocity, the ship sails west with 10 parts, and a sailor walks east in the ship with 1 part. Find the sailor's true velocity and his velocity relative to the earth.

Count east as positive; velocities along the same line add, each measured from what carries it.

True velocity: $v = 10\ 010 - 10 + 1 = 10\ 001$ parts, eastwards.

Relative to the earth, drop the earth's share: $-10 + 1 = -9$, so 9 parts westwards.

Answer: 10 001 parts east truly; 9 parts west relative to the earth

Practice

1. Keep the earth's 10 010 parts east and the ship's 10 parts west, but let the sailor walk west with 1 part. What are his true and relative velocities?

Hint: Count east as positive and add the three velocities.

Answer: True: $10\ 010 - 10 - 1 = 9\ 999$ parts east. Relative to the earth: $-10 - 1 = -11$, so 11 parts west.

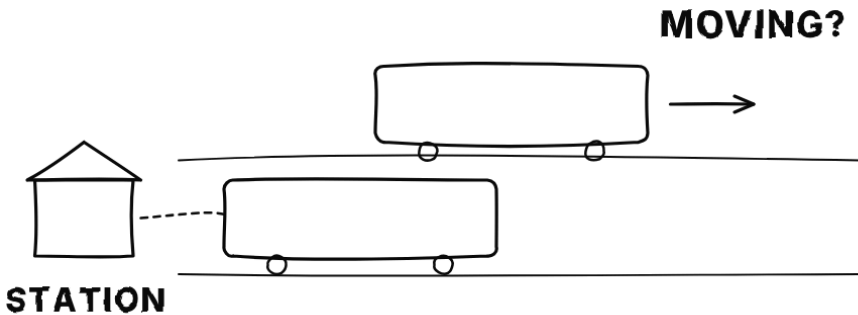
2. A train moves at 80 kilometres per hour and you walk towards the front at 5. What speed does a GPS app show, and what does it leave out?

Hint: Add velocities along the line, measured from the ground.

Answer: 85 kilometres per hour: $80 + 5$. It leaves out the earth's own motion, because it measures only relative to the ground.

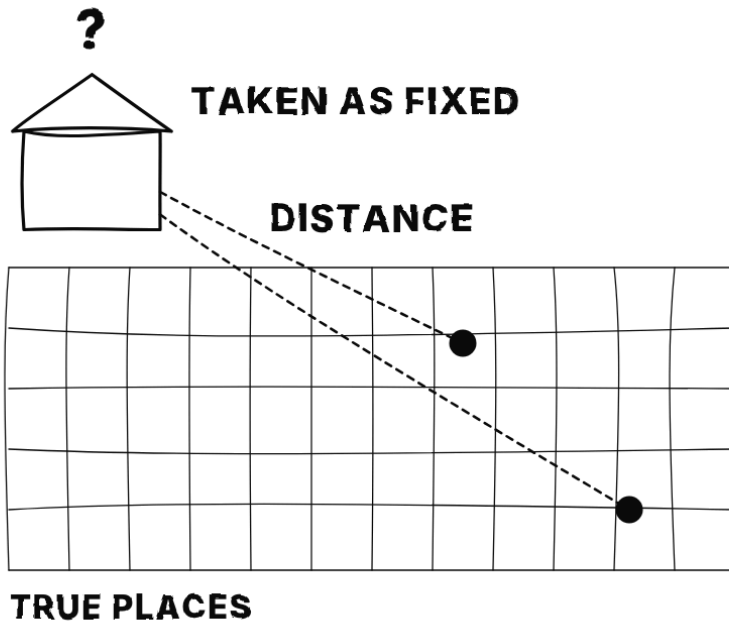
Why True Motion Cannot Be Read from Neighbours

Sit in a train at a platform while the train on the next track starts to roll. For a few seconds you would swear that you are the one moving, until you glance at the station clock tower and see that nothing has changed. Your eyes compared you with the nearest large thing, and the nearest large thing was moving. Newton asks whether any comparison with neighbours could ever settle who truly moves, and his answer is no.



Who Is Moving?

He begins with order. The parts of time keep their order forever: this moment never trades places with the next. The parts of space are just as fixed: to move a part of space out of its place would be to move it out of itself, as Newton puts it, because times and spaces are the places of themselves and of everything else. Everything sits in time in an order of succession and in space in an order of situation, so these are the true, absolute places, and passing out of them is the only true motion.

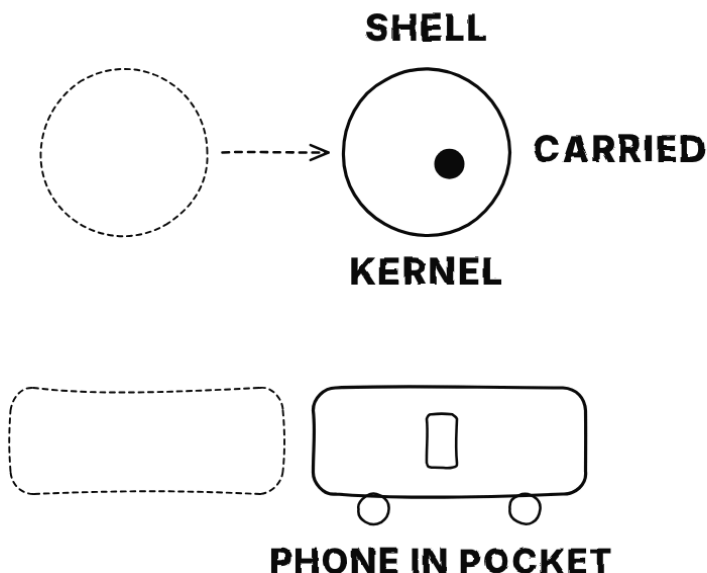


Places Measured From a Body

The trouble is that we cannot see the parts of space or tell one from another. So we pick a body we treat as unmoving, a station, the ground, the shore, and define every place by its distances and directions from that body; then we judge every motion against those

places. For everyday life this does no harm. But in natural philosophy, Newton says, we must look past our senses to the things themselves, because there may be no body anywhere truly at rest to measure from. The map on your phone does exactly this: it measures every position from the ground under your feet, taken as fixed, which works perfectly for finding a café and says nothing about how the earth itself moves.

He promises to tell true from relative motion by their properties, their causes and their effects, and starts with properties. The first property: bodies truly at rest are at rest with respect to one another. There might be a body truly at rest among the distant fixed stars or far beyond them; but from the positions of the bodies in our part of the world we can never learn whether any of them keep a fixed position towards it. So true rest cannot be found from the positions of nearby bodies.

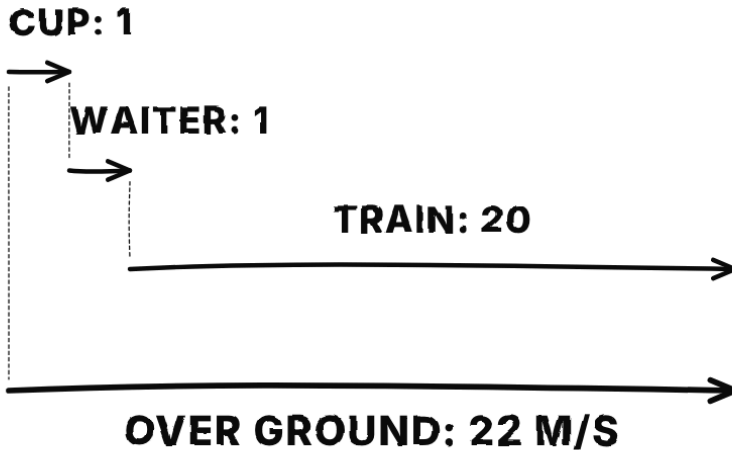


If the Shell Moves, the Kernel Moves

The second property: parts that keep fixed positions within a whole share the motion of the whole. Every part of a spinning body strains away from the axis, and the forward push of a moving body is the sum of the pushes of its parts. So if the surrounding bodies move, whatever rests among them moves with them. That is why true motion cannot be found by watching a body against things that only seem to rest: those things would have to be truly at rest. Newton's picture is a shell and its kernel: if the shell moves, the kernel moves too, as part of the whole, without ever moving away from the shell. The phone lying still in your pocket on a moving bus is the kernel; the bus is the shell, and the phone travels with it though it never shifts inside your pocket.

The third property follows closely: if a place moves, everything in it moves with it, so a body moving out of a moving place also shares the place's motion. Every motion measured from a moving place is therefore only a part of the whole, true motion. The whole is built up step by step: the body's motion within its place, plus that place's motion within its own place, and so on, until some unmoving place is reached, as with the sailor we met. In symbols, with each motion measured from the place that carries it, $v_{\text{true}} = v_1 + v_2 + v_3 + \dots$. In words: true motion is the sum of all the nested motions, down to a place that does not move. A calculation: a coffee cup slides forward on a tray at 1 metre per second, the waiter walks forward at 1 metre per second along a train, and the train runs at 20 metres per second over the ground, so relative to the ground the cup moves at $1 + 1 + 20 = 22$ metres per second, and that is still not its true motion, because the ground itself is carried along.

$$v_{true} = v_1 + v_2 + v_3 + \dots$$

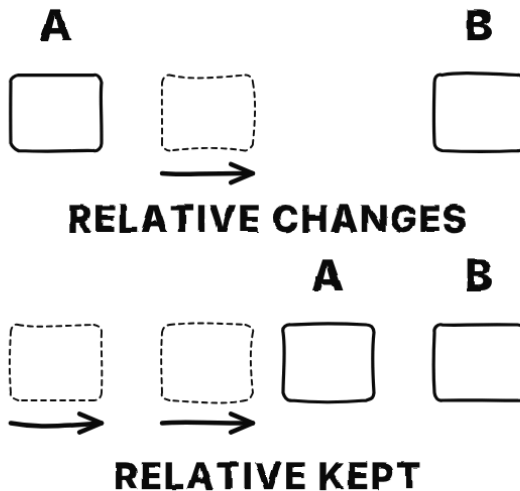


True Motion Is a Sum

So whole, true motions can only be fixed against unmoving places, which is why Newton referred true motion to unmoving places and relative motion to moving ones. And the only unmoving places are those which, from infinity to infinity, all keep the same positions towards one another; together they make up unmoving space.

Then come the causes. What separates true from relative motion is force impressed on a body. True motion is never created or changed except by a force pressed on the body itself. Relative motion can be created or changed with no force on the body at all: it is enough to push the bodies it is compared with, so that they move away and the relation changes. And the reverse holds too: a force on the body always changes its true motion, but its relative motion need not

change, if the same force is applied to the bodies it is compared with, so that their relative positions stay the same. When your bus brakes, the phone in your pocket and you slow down together; relative to you the phone does not move at all, yet its true motion has plainly changed.



Push the Other Body

So relative motion can change while true motion stays the same, and stay the same while true motion changes. True motion, Newton concludes, simply does not consist in such relations.

Remember

The parts of time and of space keep their order forever, and passing out of those true places is the only true motion. Because we cannot see the parts of space, we measure from bodies we treat as still, which works in daily life but cannot reveal true rest. Parts share the motion of their whole, as a kernel moves with its shell, and true motion is the sum of nested motions down to an unmoving place. A force on a body always changes its true motion, while its relative motion can change with no force on it, or stay the same despite one. So true motion does not consist in relations to other bodies.

Questions

1. Why can true rest not be found from the positions of nearby bodies?
 - a. Because nearby bodies are always moving
 - b. Because a body truly at rest might be far beyond the stars, and nearby positions cannot show whether bodies keep still towards it
 - c. Because positions cannot be measured
 - d. Because rest does not exist
2. What does Newton's shell and kernel picture show?
 - a. That the kernel moves only if it leaves the shell

- b. That parts keeping fixed positions within a whole share the whole's motion
 - c. That shells move faster than kernels
 - d. That only the shell has a true motion
- 3. Using $v_{\text{true}} = v_1 + v_2 + v_3 + \dots$, a cup slides at 1 m/s on a tray, the waiter walks at 1 m/s along a train, and the train runs at 20 m/s. What is the cup's speed over the ground?
 - a. 20 metres per second
 - b. 21 metres per second
 - c. 22 metres per second
 - d. 2 metres per second
- 4. How can a body's relative motion change with no force pressed on it?
 - a. It cannot: relative motion needs a force on the body
 - b. By pushing the bodies it is compared with, so the relation changes
 - c. By the passing of absolute time
 - d. By the body's own inertia

Answers

1. **b)** There might be a body truly at rest among the distant fixed stars or beyond, but from the positions of bodies in our part of the world we can never learn whether any keep a fixed position towards it.
2. **b)** If the shell moves, the kernel moves too, as part of the whole, without ever moving away from the shell.
3. **c)** Each motion is measured from the place that carries it, and nested motions add: $1 + 1 + 20 = 22$ metres per second.

4. b) Relative motion can be created or changed with no force on the body: it is enough to push the bodies it is compared with, so that they move away and the relation changes.

Worked examples

Nested motions on a train A coffee cup slides forward on a tray at 1 metre per second, the waiter walks forward along the train at 1 metre per second, and the train runs over the ground at 20 metres per second. Find the cup's speed over the ground.

True motion is built from nested motions, each measured from the place that carries it.

Add them: $v = 1 + 1 + 20$.

The result is 22 metres per second over the ground; the ground's own motion is still left out.

Answer: 22 metres per second relative to the ground

Practice

1. The waiter now walks backwards along the train at 1 metre per second while the cup still slides forward on the tray at 1 metre per second; the train runs at 20 metres per second. What is the cup's speed over the ground?

Hint: Count forward as positive and add the three nested motions.

Answer: 20 metres per second: $1 - 1 + 20 = 20$.

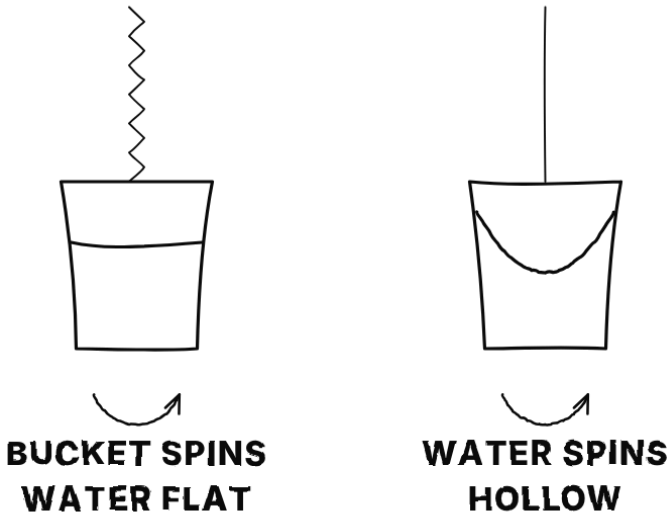
2. Explain why a force on a body always changes its true motion but need not change its relative motion.

Hint: Think of what happens if the bodies it is compared with are pushed in the same way.

Answer: True motion changes only through a force on the body itself, so a force on it always changes it. But if the same forces act on the bodies it is compared with, their relative positions stay the same, so the relative motion does not change.

The Bucket, the Two Globes and the Purpose of the Book

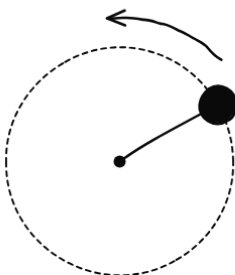
Hang a small bucket of water from a long string, twist the string many times, hold the bucket still, then let it go. As the string unwinds, the bucket spins faster and faster, yet for a while the water inside stays flat. Then, little by little, the water starts to turn, its middle sinks, and it climbs the sides into a hollow, until water and bucket spin together. Newton made this experiment himself, and it is the effect that finally separates true motion from relative motion.



The effects that tell absolute from relative motion, he says, are the forces of receding from the axis of circular motion, the outward strain of anything that spins. A purely relative circling produces no such strain; a true circling produces more or less of it, according to how much circular motion there is. Today we describe that strain with a formula for the force needed to keep a mass m moving at speed v in a circle of radius r : $F = mv^2/r$. In words: the faster something circles, and the tighter its circle, the harder it pulls outwards, growing with the square of the speed. The intuition is the sling we met at the start: whirl the stone twice as fast and the sling pulls four times as hard. A calculation: a 1-kilogram ball whirled at 2 metres per second on a 1-metre string pulls with $1 \times 2^2 / 1 = 4$ newtons; at 4 metres per second it pulls with $1 \times 4^2 / 1 = 16$ newtons.

$$F = \frac{m v^2}{r}$$

F: PULL, M: MASS
V: SPEED, R: RADIUS

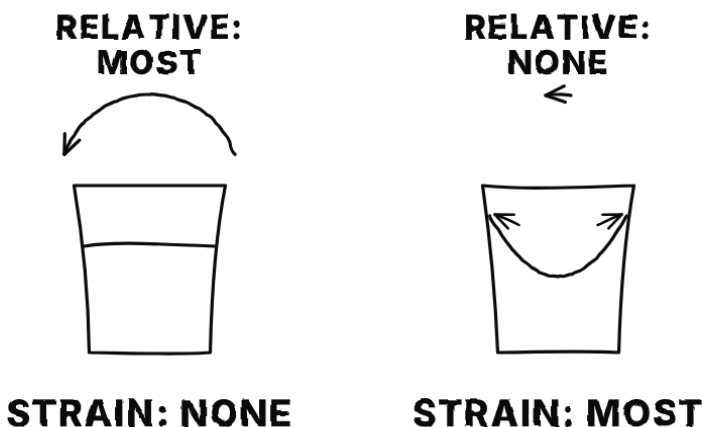


$$V = 2: F = 4 \text{ N}$$

$$V = 4: F = 16 \text{ N}$$

The Outward Pull: $F = mv^2/r$

Now read the bucket with that in mind. At the start the water moved fastest relative to the bucket, since the bucket was turning around still water, yet its surface stayed flat: no outward strain, so no true circling yet. Later, as the water caught up and its motion relative to the bucket shrank, it climbed the sides, and climbed highest exactly when it spun along with the bucket and was at rest relative to it. The rise of the water follows its true circling, which here runs directly opposite to its relative motion, and the strain can even be used to measure it. For a liquid spinning as one piece, today's formula for the rise at the rim above the centre is $h = \omega^2 r^2 / (2g)$, where ω , the Greek letter omega, is the turning rate in radians per second and g is gravity's 9.8. Spinning 2 turns per second, ω is about 12.6, so in a bucket of radius 0.1 metre the rim rises about $12.6^2 \times 0.1^2 / (2 \times 9.8) \approx 0.08$ metres, some 8 centimetres.



Strain Follows True Spinning

So the strain does not depend on any motion of the water relative to the things around it, and true circular motion cannot be defined by such motion. Each spinning body has only one true circular motion, matched by one outward strain, its proper effect. But it has countless relative motions, one for every body it might be compared with, and like any mere relations, they have no real effect except as they share in that one true motion. This settles an old picture of the heavens, in which planets rest within turning spheres beneath the fixed stars and are carried round by them: those planets would still truly move, since they change positions towards one another, which bodies truly at rest never do, and as parts of a spinning whole they strain away from its axis. The spin cycle of a washing machine shows the strain at full

strength: the drum's true spinning presses the clothes so hard against the wall that the water is flung out through the holes.

Try it yourself

Stir a cup of tea briskly and take the spoon out. While the tea spins, its surface dips in the middle and rises at the rim. Watch it from your chair, then from the other side of the table: the hollow looks exactly the same from both, because it belongs to the tea's true spinning and not to your point of view.

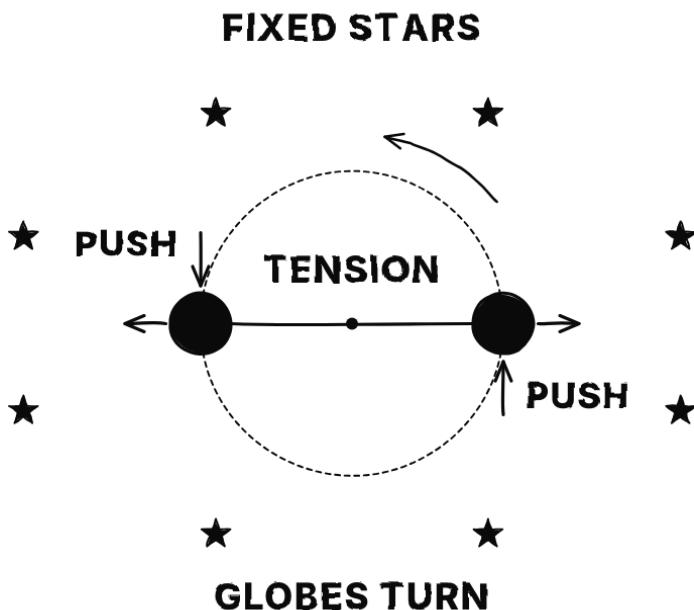
From all this Newton draws a lesson about words. Relative quantities are not the quantities whose names they carry, but the visible measures of them, accurate or not, which people commonly use instead. If words mean what common use makes them mean, then time, space, place and motion should be taken to mean these measures, and using them for the true quantities is unusual and purely mathematical. So those who read the true quantities into the sacred writings strain them, and those who mix up real quantities with their relations and common measures spoil the purity of mathematical and philosophical truth.

QUANTITIES $\neq$ MEASURES



Quantities and Their Measures

Finding the true motions of particular bodies is very hard, Newton admits, because the parts of unmoving space can never be seen. But it is not hopeless, because there are two kinds of clue: the apparent motions, which are the differences between true motions, and the forces, which are their causes and effects. His example is a thought experiment. Two globes, tied together by a cord and kept at a fixed distance, spin about their common centre of gravity in a vast empty space. From the tension of the cord we can find their outward strain, and from that, how much circular motion they have. With $F = mv^2/r$ in hand: if each globe has 1 kilogram, sits 1 metre from the centre, and the cord pulls with 4 newtons, then $v^2 = Fr/m = 4$, so each globe moves at 2 metres per second, found without looking at anything outside.



Two Globes on a Cord

Next, press equal forces at the same moment on matching faces of the two globes. If the cord gets tighter, those forces sped the globes up; if it gets slacker, they slowed them down. That tells us which faces trail behind in the circling, and so which faces lead, and therefore which way the globes are turning. So both the amount and the direction of the circular motion can be found even in an immense empty space with nothing to compare the globes with.

Now add some far-off bodies that always keep the same positions towards one another, as the fixed stars do. Watching the globes move against them, we could not tell whether the globes were turning or the far bodies were. But if the cord's tension is exactly the tension the globes' motion requires, we may conclude that the globes move and

the bodies are at rest, and then, from the globes' passage among the bodies, we can find the direction of their motion. A phone's gyroscope works in this spirit: it senses its own true turning from forces inside the chip, without looking at anything outside, which is how your screen knows you have rotated it in a dark room.

How to gather true motions from their causes, effects and apparent differences, and, the other way round, how to learn the causes and effects from the motions, true or apparent, is what the rest of the book explains. For that purpose, Newton says, he wrote it. With these definitions and this scholium, the stage is set: matter, motion and force are measured, true motion is separated from what merely appears, and the laws of motion can begin.

Remember

True circular motion reveals itself by an outward strain, which grows as mv^2/r , while merely relative circling has none. In Newton's bucket the water climbs the sides only as its true circling grows, highest when it is at rest relative to the bucket, so true rotation is not a relation to surrounding bodies. Everyday words like time and motion name our measures, and real quantities must not be confused with those measures. Two globes on a cord show how forces reveal motion: the cord's tension gives the amount of circling, extra pushes give its direction, even with nothing outside to compare with. Finding true motions from causes, effects and appearances, and causes from motions, is the purpose of the whole book.

Questions

1. Using $F = mv^2/r$, what happens to the pull of a whirled ball when its speed doubles on the same string?
 - a. It doubles
 - b. It stays the same
 - c. It becomes four times as large
 - d. It halves
2. Why do everyday words like time and motion, according to Newton, properly mean their measures?
 - a. Because the true quantities do not exist
 - b. Because the meaning of words is set by their common use
 - c. Because measures are always accurate
 - d. Because scripture defines them so
3. Two globes on a cord spin in empty space. Equal pushes on matching faces make the cord tighter. What does that tell us?
 - a. The pushes slowed the globes down
 - b. The pushes sped the globes up, so those faces trail in the circling
 - c. The globes stopped spinning
 - d. Nothing, without stars to compare
4. With far-off bodies around, how can we decide that the globes, not the bodies, are moving?
 - a. By watching the globes pass among the bodies

- b. By checking that the cord's tension is just what the globes' motion requires
- c. By weighing the far bodies
- d. It can never be decided

Answers

1. **c)** The pull grows with the square of the speed: twice the speed gives four times the pull, as 4 newtons at 2 metres per second become 16 at 4.
2. **b)** If words mean what common use makes them mean, then time, space, place and motion should be taken to mean their measures.
3. **b)** If the cord gets tighter, those forces sped the globes up, which tells us which faces trail behind and so which way the globes turn.
4. **b)** If the cord's tension is exactly the tension the globes' motion requires, we may conclude that the globes move and the bodies are at rest.

Worked examples

Speed of the globes from the cord Two globes of 1 kilogram each circle 1 metre from their common centre, and the cord pulls on each with 4 newtons. How fast does each globe move?

The cord supplies the inward force $F = mv^2/r$ that keeps each globe circling.

Solve for the speed: $v^2 = Fr/m = 4 \times 1 / 1 = 4$.

Take the square root: $v = 2$.

Answer: 2 metres per second

Practice

1. Using $F = mv^2/r$, how hard does a 1-kilogram ball pull on a 1-metre string when whirled at 3 metres per second?

Hint: Square the speed, multiply by the mass and divide by the radius.

Answer: 9 newtons: $1 \times 3^2 / 1 = 9$.

2. In the globe experiment, the cord now pulls with 9 newtons on each 1-kilogram globe at 1 metre from the centre. How fast does each globe move?

Hint: Use $v^2 = Fr/m$ and take the square root.

Answer: 3 metres per second: $v^2 = 9 \times 1 / 1 = 9$, so $v = 3$.

3. Explain why the water in Newton's bucket climbs the sides only when it truly spins.

Hint: Compare the flat surface at the start with the hollow at the end.

Answer: At the start the water moves fastest relative to the bucket but stays flat, so it has no outward strain and is not yet truly circling. As its own true circling grows, the strain grows and it climbs, highest when it is at rest relative to the bucket.