

Collected Papers of Srinivasa Ramanujan



Srinivasa Ramanujan · 1927

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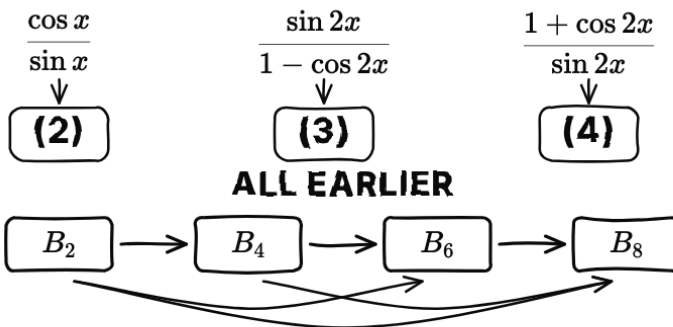
Some properties of Bernoulli's numbers

Journal of the Indian Mathematical Society, volume three, 1911, pages 219 to 234.

Section 1.

Let the well-known expansion of $x \cot x$, see Edwards' Differential Calculus, section 149, be written in the form $x \cot x = 1 - (B_2/2!)(2x)^2 - (B_4/4!)(2x)^4 - (B_6/6!)(2x)^6 - \dots$, which is equation one, from which we infer that B_0 may be supposed to be minus one.

Now $\cot x = \cos x / \sin x = (1 - x^2/2! + x^4/4! - x^6/6! + \dots) / (x - x^3/3! + x^5/5! - x^7/7! + \dots)$. It also equals $\sin 2x / (1 - \cos 2x) = (2x/1! - (2x)^3/3! + (2x)^5/5! - \dots) / ((2x)^2/2! - (2x)^4/4! + (2x)^6/6! - \dots)$. And it equals $(1 + \cos 2x) / \sin 2x = (2 - (2x)^2/2! + (2x)^4/4! - \dots) / (2x - (2x)^3/3! + (2x)^5/5! - \dots)$.



$$(x \cot x)^2 = -x^2 \left(1 + \frac{d \cot x}{dx} \right)$$

$$\frac{1}{2}(n+1)B_n = c_2 B_2 B_{n-2} + c_4 B_4 B_{n-4} + \cdots \quad (5)$$

$$\frac{d \tan x}{dx} = 1 + \tan^2 x$$

Three ratios, three equations

Multiplying both sides in each of the above three relations by the denominator of the right-hand side and equating the coefficients of x^n on both sides, we can write the results thus. The first is $c_1 B_{n-1}/2 - c_3 B_{n-3}/2^3 + c_5 B_{n-5}/2^5 - \cdots + (-1)^{((n-1)/2)} B_0/2^n + (n/2^n) (-1)^{((n-1)/2)} = 0$, equation two, where n is any odd integer. The second is $c_2 B_{n-2} - c_4 B_{n-4} + c_6 B_{n-6} - \cdots + (-1)^{((n-2)/2)} B_0 + (n/2) (-1)^{((n-2)/2)} = 0$, equation three, where n is any even integer. The third is $c_1 B_{n-1} - c_3 B_{n-3} + c_5 B_{n-5} - \cdots + (-1)^{((n-1)/2)} B_0 + (n/2) (-1)^{((n-1)/2)} = 0$, equation four, where n is any odd integer greater than unity.

From any one of equations two, three and four we can calculate the B 's. But as n becomes greater and greater the calculation will get

tedious. So we shall try to find simpler methods.

Section 2.

We know that $(x \cot x)^2 = -x^2(1 + d \cot x / dx)$. Using equation one and equating the coefficients of x^n on both sides, and simplifying, we have $\frac{1}{2}(n + 1) B_n = c_2 B_2 B_{n-2} + c_4 B_4 B_{n-4} + c_6 B_6 B_{n-6} + \cdots$, the last term being $c_{(n/2 - 1)} B_{(n/2 - 1)} B_{(n/2 + 1)}$ or $\frac{1}{2} c_{(n/2)} (B_{(n/2)})^2$ according as $\frac{1}{2}n$ is odd or even; this is result five.

A similar result can be obtained by equating the coefficients of x^n in the identity $d \tan x / dx = 1 + \tan^2 x$.

Section 3.

COT	B₀	B₂	B₄	B₆	B₈	B₁₀
COTH	+	−	+	−	+	−
SUM						
DIFF						

$\frac{n}{2}$ terms

$\frac{n}{4}$ terms

Every second term cancels

Again, $-1/2x(\cot 1/2x + \coth 1/2x) = -1/2x(\cot 1/2x + i \cot 1/2ix) = 2(B_0 + B_4 x^4/4! + B_8 x^8/8! + \dots)$, by using equation one. The expression may also be written $-1/2x (\cos 1/2x \sin 1/2ix + i \sin 1/2x \cos 1/2ix) / (\sin 1/2x \sin 1/2ix)$, which is $-1/2x ((1 + i) \sin 1/2x(1 + i) - (1 - i) \sin 1/2x(1 - i)) / (\cos 1/2x(1 - i) - \cos 1/2x(1 + i))$, which is $-x (x/1! - x^5/(2^2 \cdot 5!) + x^9/(2^4 \cdot 9!) - \dots) / (x^2/2! - x^6/(2^2 \cdot 6!) + x^{10}/(2^4 \cdot 10!) - \dots)$, by expanding the numerator and the denominator, and simplifying by De Moivre's theorem.

Hence $2(B_0 + B_4 x^4/4! + B_8 x^8/8! + \dots) = -x (x/1! - x^5/(2^2 \cdot 5!) + \dots) / (x^2/2! - x^6/(2^2 \cdot 6!) + \dots)$, which is equation six.

Similarly, $-1/2x(\cot 1/2x - \coth 1/2x) = 2(B_2 x^2/2! + B_6 x^6/6! + B_{10} x^{10}/10! + \dots)$. This equals $x (1/2(1 - i) \sin 1/2x(1 + i) - 1/2(1 + i) \sin 1/2x(1 - i)) / (\cos 1/2x(1 + i) - \cos 1/2x(1 - i))$, which is $x (x^3/(2 \cdot 3!) - x^7/(2^3 \cdot 7!) + x^{11}/(2^5 \cdot 11!) + \dots) / (x^2/2! - x^6/(2^2 \cdot 6!) + x^{10}/(2^4 \cdot 10!) - \dots)$, which is equation seven.

Proceeding as in section 1 we have, if n is an even integer greater than 2, $c_2 B_{n-2}/2 - c_6 B_{n-6}/2^3 + c_{10} B_{n-10}/2^5 - \dots + (n/2^{((n+2)/2)}) (-1)^{(n/4)} \text{ or } (n/2^{((n+2)/2)}) (-1)^{((n-2)/4)} = 0$, equation eight, according as n or n minus 2 is a multiple of 4.

Analogous results can be obtained from $\tan 1/2x \pm \tanh 1/2x$.

In equations two, three and four there are $1/2n$ terms, while in five and eight there are $1/4n$ or $1/4(n - 2)$ terms. Thus B_n can be found from only half of the previous B 's.

Questions

1. Why may B_0 be supposed to be -1 in equation (1)?
 - a. Because every Bernoulli number with an even index is negative
 - b. Because the constant term 1 of $x \cot x$ then fits the same pattern as the other terms, $-B_0$ times $(2x)^0$ over $0!$
 - c. Because $\cot x$ is negative near zero
 - d. Because B_0 must cancel the term with B_2

2. How does Ramanujan turn the three expressions for $\cot x$ into equations (2), (3) and (4)?
 - a. By integrating both sides from 0 to x
 - b. By setting x equal to small numbers and solving
 - c. By multiplying each relation by its right-hand denominator and equating the coefficients of x^n on both sides
 - d. By taking logarithms of both sides

3. Why does Ramanujan look for other methods after equations (2) to (4)?
 - a. Because those equations are false for large n
 - b. Because as n grows the calculation becomes tedious
 - c. Because they give only the odd-index B 's
 - d. Because they need the value of π

4. What do the relations (5) and (8) gain over (2), (3) and (4)?
 - a. They need only about a quarter of n terms instead of half of n , so B_n comes from only half of the previous B 's
 - b. They give B_n without any earlier B 's
 - c. They work only for odd n
 - d. They give the denominators of the B 's directly

Answers

1. **b)** Equation (1) writes every term as minus B over a factorial times a power of $2x$; the first term, 1, has that form only if B_0 is -1 .
2. **c)** Multiplying by the denominator leaves two power series that must agree term by term; equating the coefficients of x^n gives one relation among the B 's.
3. **b)** Any of the three equations gives the B 's, but as n becomes greater and greater the calculation gets tedious.
4. **a)** In (2), (3) and (4) there are half of n terms, in (5) and (8) a quarter of n or a quarter of n minus 2, so only half of the previous B 's are needed.

Worked examples

The first two B 's from equation (3) Use equation (3), with $B_0 = -1$, for $n = 4$ and $n = 6$ to find B_2 and B_4 . Here c_k is the number of ways to choose k things out of n .

For $n = 4$ the terms are $c_2 B_2$, then the B_0 term with sign $(-1)^{((4-2)/2)} = -1$, then $(4/2)$ times the same sign.

With $c_2 = 6$ this reads $6B_2 - B_0 - 2 = 0$, and since $B_0 = -1$ that is $6B_2 + 1 - 2 = 0$, so $B_2 = 1/6$.

For $n = 6$ the coefficients are $c_2 = 15$ and $c_4 = 15$, and the sign of the B_0 term is $(-1)^2 = +1$.

So $15B_4 - 15B_2 + B_0 + 3 = 0$, that is $15B_4 - 15/6 - 1 + 3 = 0$.

Hence $15B_4 = 1/2$ and $B_4 = 1/30$.

Answer: $B_2 = 1/6$ and $B_4 = 1/30$

Practice

1. Put $n = 8$ into equation (3) and find B_6 , using $B_2 = 1/6$, $B_4 = 1/30$ and $B_0 = -1$.

Hint: The coefficients are $c_2 = 28$, $c_4 = 70$, $c_6 = 28$, and the sign $(-1)^{((8-2)/2)}$ is -1 .

Answer: $B_6 = 1/42$. The equation is $28B_6 - 70/30 + 28/6 - B_0 - 4 = 0$, that is $28B_6 - 7/3 + 14/3 + 1 - 4 = 0$, so $28B_6 = 2/3$.

2. Use result (5) with $n = 4$ to find B_4 from $B_2 = 1/6$.

Hint: Half of 4 is 2, which is even, so the only term is $\frac{1}{2} c_2 (B_2)^2$, with $c_2 = 6$.

Answer: $B_4 = 1/30$. Result (5) gives $\frac{1}{2} \times 6 \times B_4 = \frac{1}{2} \times 6 \times (1/6)^2 = 1/12$, so $B_4 = 2/60 = 1/30$.

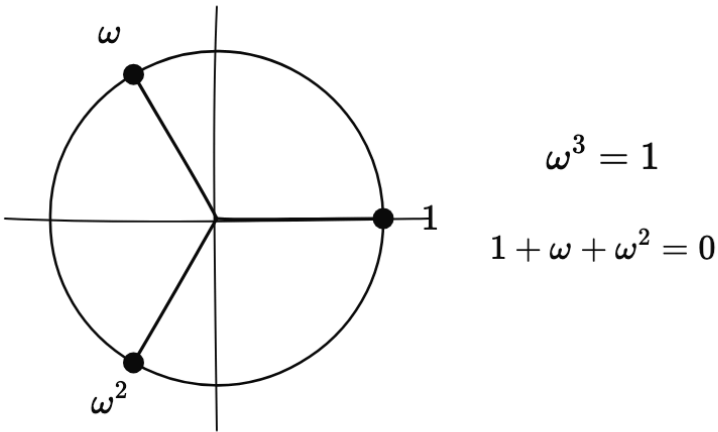
3. Check equation (1) numerically for $x = 0.1$ using its first three terms, and compare with $0.1 \cot 0.1 = 0.9966644$.

Hint: The second term is $(B_2/2!)(2x)^2$ with $B_2 = 1/6$; the third uses $B_4 = 1/30$ and $4! = 24$.

Answer: $1 - 0.04/12 - 0.0016/720 = 1 - 0.0033333 - 0.0000022 = 0.9966645$, agreeing with 0.9966644 to the sixth decimal; the next term is smaller still.

Section 4

A still simpler method can be deduced from the following identities.



$$-\frac{1}{2}x \left(\cot \frac{1}{2}x + \omega \cot \frac{1}{2}x\omega + \omega^2 \cot \frac{1}{2}x\omega^2 \right)$$

The three cube roots of unity

If $1, \omega, \omega^2$ be the three cube roots of unity, then $4 \sin x \sin x\omega \sin x\omega^2 = -(\sin 2x + \sin 2x\omega + \sin 2x\omega^2)$, as may easily be verified.

By logarithmic differentiation, we have $\cot x + \omega \cot x\omega + \omega^2 \cot x\omega^2 = 2(\cos 2x + \omega \cos 2x\omega + \omega^2 \cos 2x\omega^2) / (\sin 2x + \sin 2x\omega + \sin 2x\omega^2)$.

Writing $\frac{1}{2}x$ for x , $-\frac{1}{2}x(\cot \frac{1}{2}x + \omega \cot \frac{1}{2}x\omega + \omega^2 \cot \frac{1}{2}x\omega^2) = -x(\cos x + \omega \cos x\omega + \omega^2 \cos x\omega^2) / (\sin x + \sin x\omega + \sin x\omega^2)$, and, proceeding as in section 3, we get $3(B_0 + B_6 x^6/6! + B_{12} x^{12}/12! + \dots) = -x(x^2/2! - x^8/8! + x^{14}/14! - \dots) / (x^3/3! - x^9/9! + x^{15}/15! - \dots)$, which is equation nine.

Again, $\cot \frac{1}{2}x\omega - \cot \frac{1}{2}x\omega^2 = (\cos x\omega^2 - \cos x\omega) / (2 \sin \frac{1}{2}x \sin \frac{1}{2}x\omega \sin \frac{1}{2}x\omega^2) = 2(\cos x\omega - \cos x\omega^2) / (\sin x + \sin x\omega + \sin x\omega^2)$.

Multiplying both sides by $-\frac{1}{2}x(\omega^2 - \omega)$ and adding to the corresponding sides of the previous result, we have $-\frac{1}{2}x(\cot \frac{1}{2}x + \omega^2 \cot \frac{1}{2}x\omega + \omega \cot \frac{1}{2}x\omega^2) = -x(\cos x + \omega^2 \cos x\omega + \omega \cos x\omega^2) / (\sin x + \sin x\omega + \sin x\omega^2)$.

Hence, as before, $3(B_2 x^2/2! + B_8 x^8/8! + B_{14} x^{14}/14! + \dots) = x(x^4/4! - x^{10}/10! + x^{16}/16! - \dots) / (x^3/3! - x^9/9! + x^{15}/15! - \dots)$, which is equation ten.

Similarly, $-x(\cot \frac{1}{2}x + \cot \frac{1}{2}x\omega + \cot \frac{1}{2}x\omega^2) = x(\cos x + \cos x\omega + \cos x\omega^2 - 3) / (\sin x + \sin x\omega + \sin x\omega^2)$, and therefore $6(B_4 x^4/4! + B_{10} x^{10}/10! + B_{16} x^{16}/16! + \dots) = x(x^6/6! - x^{12}/12! + x^{18}/18! - \dots) / (x^3/3! - x^9/9! + x^{15}/15! - \dots)$, which is equation eleven.

Multiplying up and equating coefficients in equations nine, ten and eleven as usual, we have $c_3 B_{n-3} - c_9 B_{n-9} + c_{15} B_{n-15} - \dots = 0$, equation twelve, the last term being $\frac{1}{6}n(-1)^{((n-1)/3)}$, $\frac{1}{3}n(-1)^{((n+1)/3)}$, or $\frac{1}{3}n(-1)^{((n-3)/3)}$.

Again, dividing both sides in equation ten by x and differentiating, we have $3(B_2/2! + 7B_8 x^6/8! + 13B_{14} x^{12}/14! + \dots) = d/dx$ of $(x^4/4! - x^{10}/10! + x^{16}/16! - \dots) / (x^3/3! - x^9/9! + x^{15}/15! - \dots)$, which equals

$$1 - (x^2/2! - x^8/8! + x^{14}/14! - \dots)(x^4/4! - x^{10}/10! + x^{16}/16! - \dots) / (x^3/3! - x^9/9! + x^{15}/15! - \dots)^2.$$

Hence by equations nine and ten, $3(B_2 x^2/2! + 7B_8 x^8/8! + 13B_{14} x^{14}/14! + \dots) = x^2 + 9(B_0 + B_6 x^6/6! + B_{12} x^{12}/12! + \dots)(B_2 x^2/2! + B_8 x^8/8! + B_{14} x^{14}/14! + \dots)$.

Equating the coefficients of x^n we have, if n is greater than 2 and n minus 2 is a multiple of 6, $\frac{1}{3}(n+2) B_n = c_6 B_{n-6} + c_{12} B_{n-12} + c_{18} B_{n-18} + \dots$, which is equation thirteen.

From equation twelve the B's can be calculated very quickly and equation thirteen may prove useful in checking the calculations. The number of terms is one-third of that in equation four; thus B_{24} is found from B_{18} , B_{12} and B_6 .

Questions

1. What does the use of the three cube roots of unity achieve in section 4?
 - a. It makes the B's integers
 - b. It separates the B's into three classes spaced six apart, so each equation keeps only every sixth B
 - c. It removes the need for coefficient comparison
 - d. It gives the odd-index B's
2. Which equation does Ramanujan say computes the B's very quickly, and which is for checking?
 - a. Equation thirteen computes them, equation twelve checks

- b. Equation twelve computes them, equation thirteen is useful in checking
 - c. Both only check earlier results
 - d. Equation four computes them, equation twelve checks
3. From which earlier B's is B_{24} found with the new method?
- a. From all of B_2 to B_{22}
 - b. From B_{18} , B_{12} and B_6
 - c. From B_{22} alone
 - d. From B_{12} alone

Answers

1. **b)** Equations nine, ten and eleven keep B_0 , B_6 , B_{12} ...; B_2 , B_8 , B_{14} ...; and B_4 , B_{10} , B_{16} ... respectively.
2. **b)** From equation twelve the B's can be calculated very quickly and equation thirteen may prove useful in checking the calculations.
3. **b)** The number of terms is one third of that in equation four; thus B_{24} is found from B_{18} , B_{12} and B_6 .

Worked examples

Checking equation thirteen for $n = 8$ Equation thirteen holds when $n - 2$ is a multiple of 6. Use it with $n = 8$, $B_2 = 1/6$ and $B_6 = 1/42$ to find B_8 .

For $n = 8$ only the first term on the right is present: $c_6 B_2 B_6$, where c_6 is the number of ways to choose 6 of 8, which is 28.

The right side is $28 \times 1/6 \times 1/42 = 28/252 = 1/9$.

The left side is $1/3(8 + 2) B_8 = (10/3) B_8$.

So $B_8 = (1/9) \times (3/10) = 1/30$.

Answer: $B_8 = 1/30$

Practice

1. Check the leading term of equation nine: for very small x , what does the right side $-x (x^2/2! - \cdots) / (x^3/3! - \cdots)$ approach, and what does that say about B_0 ?

Hint: Keep only the first term of the top and of the bottom.

Answer: It approaches $-x (x^2/2) / (x^3/6) = -3$, and the left side approaches $3B_0$, so $B_0 = -1$, as in section 1.

2. Check the leading term of equation ten in the same way and find B_2 .

Hint: The right side starts with $x (x^4/24) / (x^3/6)$; the left side with $3 B_2 x^2/2$.

Answer: The right side is $x^2/4$, so $3B_2/2 = 1/4$ and $B_2 = 1/6$.

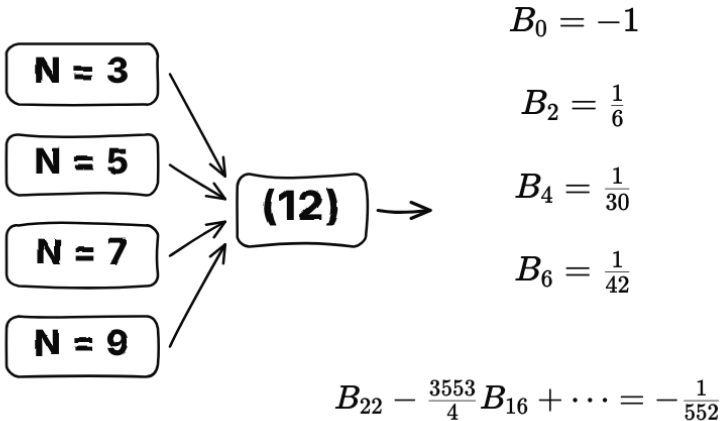
3. Check the leading term of equation eleven and find B_4 .

Hint: The right side starts with $x (x^6/720) / (x^3/6)$; the left side with $6 B_4 x^4/24$.

Answer: The right side is $x^4/120$ and the left side $B_4 x^4/4$, so $B_4 = 1/30$.

Sections 5 to 7

Section 5.

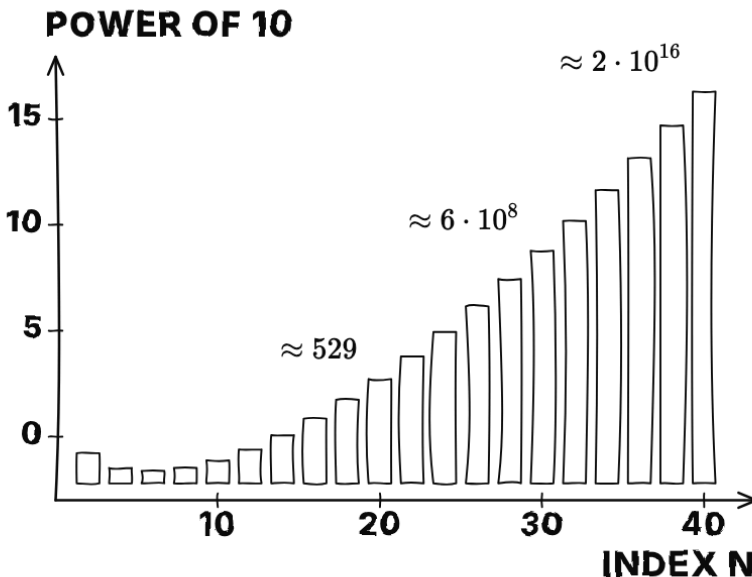


Feeding $n = 3, 5, 7, \dots$ into (12)

We shall see later on how the B's can be obtained from their properties only. But to know these properties, it will be convenient to calculate a few B's by substituting 3, 5, 7, 9, and so on, for n in succession in equation twelve. Thus $B_0 = -1$; $B_2 = 1/6$; $B_4 = 1/30$; $B_6 = 1/42$; $B_8 - \frac{1}{3}B_2 = -1/45$. Next, $B_{10} - (5/2)B_4 = -1/132$; $B_{12} - 11B_6 = -4/455$; $B_{14} - (143/4)B_8 + B_2/5 = 1/120$. Next, $B_{16} - (286/3)B_{10} + 4B_4 = 1/306$; $B_{18} - 221B_{12} + (204/5)B_6 = 3/665$. Then $B_{20} -$

$(3230/7)B_{14} + (1938/7)B_8 - B_2/7 = -1/231$. And $B_{22} - (3553/4)B_{16} + (7106/5)B_{10} - (11/2)B_4 = -1/552$, and so on.

Hence we have finally the following values: $B_2 = 1/6$; $B_4 = 1/30$; $B_6 = 1/42$; $B_8 = 1/30$; $B_{10} = 5/66$; $B_{12} = 691/2730$. Then $B_{14} = 7/6$; $B_{16} = 3617/510$; $B_{18} = 43867/798$; $B_{20} = 174611/330$; $B_{22} = 854513/138$. Then $B_{24} = 236364091/2730$; $B_{26} = 8553103/6$; $B_{28} = 23749461029/870$. Then $B_{30} = 8615841276005/14322$; $B_{32} = 7709321041217/510$; $B_{34} = 2577687858367/6$. Then $B_{36} = 26315271553053477373/1919190$; $B_{38} = 2929993913841559/6$. And $B_{40} = 261082718496449122051/13530$, and so on, with $B_{\infty} = \infty$.



Small at first, then enormous

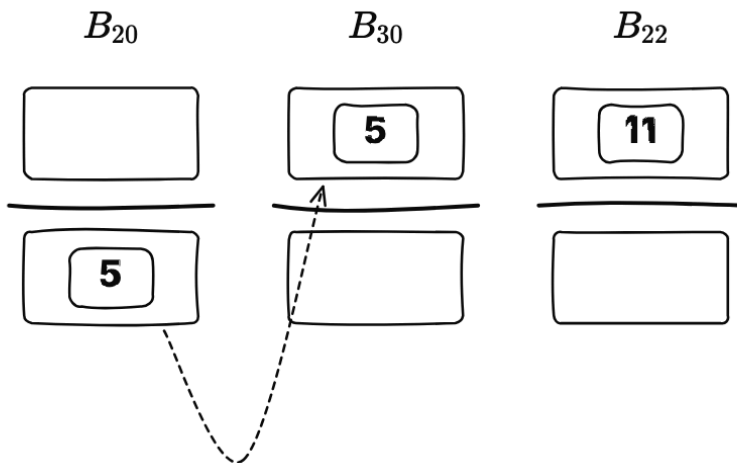
Section 6.

It will be observed that, if n is even but not equal to zero, the following hold. First, B_n is a fraction and the numerator of B_n/n in its lowest terms is a prime number; this is statement fourteen. Second, the denominator of B_n contains each of the factors 2 and 3 once and only once; this is statement fifteen. Third, $2^n(2^n - 1)B_n/n$ is an integer and consequently $2(2^n - 1)B_n$ is an odd integer; this is statement sixteen.

From statement sixteen it can easily be shewn that the denominator of $2(2^n - 1)B_n/n$ in its lowest terms is the greatest power of 2 which divides n ; and consequently, if n is not a multiple of 4, then $4(2^n - 1)B_n/n$ is an odd integer; this is statement seventeen.

It follows from statement fourteen that the numerator of B_n in its lowest terms is divisible by the greatest measure of n prime to the denominator, and the quotient is a prime number; this is statement eighteen.

Examples. First, 2 and 3 are the only prime factors of 12, 24 and 36, and they are found in the denominators of B_{12} , B_{24} and B_{36} , and their numerators are prime numbers.



A prime goes up or down

Second, 11 is not found in the denominator of B_{22} , and hence its numerator is divisible by 11; similarly, the numerators of B_{26} , B_{34} and B_{38} are divisible by 13, 17 and 19 respectively, and the quotients in all cases are prime numbers.

Third, 5 is found in the denominator of B_{20} and not in that of B_{30} , and consequently the numerator of B_{30} is divisible by 5 while that of B_{20} is a prime number. Thus we may say that if a prime number appearing in n is not found in the denominator it will appear in the numerator, and vice versa.

Section 7.

$$B_{24} = \frac{236364091}{2730}$$

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 6 & 8 & 12 & 24 \\ \downarrow +1 & & & & & & & & \\ 2 & 3 & \cancel{4} & 5 & 7 & \cancel{9} & 13 & \cancel{25} & \end{array}$$

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$$

$$B_2 : 6$$

$$B_{20} : 330$$

$$B_{12} : 2730$$

$$B_{30} : 14322$$

Building the denominator of B24

Next, let us consider the denominators.

All the denominators are divisible by 6; those of B₄, B₈, B₁₂, and so on, by 5; those of B₆, B₁₂, B₁₈, and so on, by 7; those of B₁₀, B₂₀, B₃₀, and so on, by 11; but those of B₈, B₁₆, B₂₄, and so on, are not divisible by 9; and those of B₁₄, B₂₈, and so on, are not divisible by 15. Hence we may infer that the denominator of B_n is the continued product of prime numbers which are the next numbers, in the natural order, to the factors of n, including unity and the number itself; this is statement nineteen.

As an example take the denominator of B₂₄. Write all the factors of 24, namely 1, 2, 3, 4, 6, 8, 12 and 24. The next numbers to these are 2,

3, 4, 5, 7, 9, 13 and 25. Strike out the composite numbers and we have the prime numbers 2, 3, 5, 7 and 13. And the denominator of B_{24} is the product of 2, 3, 5, 7 and 13, that is, 2730.

It is unnecessary to write the odd factors of n except unity, as the next numbers to these are even and hence composite.

The following are some further examples, listing the even factors of n with unity, and the denominator. For B_2 : 1 and 2, giving 2 times 3, which is 6. For B_6 : 1, 2 and 6, giving 2 times 3 times 7, which is 42. For B_{12} : 1, 2, 4, 6 and 12, giving 2 times 3 times 5 times 7 times 13, which is 2730. For B_{20} : 1, 2, 4, 10 and 20, giving 2 times 3 times 5 times 11, which is 330. For B_{30} : 1, 2, 6, 10 and 30, giving 2 times 3 times 7 times 11 times 31, which is 14322. For B_{42} : 1, 2, 6, 14 and 42, giving 2 times 3 times 7 times 43, which is 1806. For B_{56} : 1, 2, 4, 8, 14, 28 and 56, giving 2 times 3 times 5 times 29, which is 870. For B_{72} : 1, 2, 4, 6, 8, 12, 18, 24, 36 and 72, giving 2 times 3 times 5 times 7 times 13 times 19 times 37 times 73, which is 140100870. For B_{90} : 1, 2, 6, 10, 18, 30 and 90, giving 2 times 3 times 7 times 11 times 19 times 31, which is 272118. For B_{110} : 1, 2, 10, 22 and 110, giving 2 times 3 times 11 times 23, which is 1518.

Questions

1. Why does Ramanujan calculate a table of B 's in section 5?
 - a. To show that equation twelve is wrong
 - b. Because to know the properties of the B 's it is convenient to have a few values first
 - c. Because the B 's cannot be obtained in any other way

- d. To find the value of π
2. According to statement fifteen, what can be said about the denominator of B_n ?
 - a. It is always a prime
 - b. It contains each of the factors 2 and 3 once and only once
 - c. It is always a power of 2
 - d. It equals $n + 1$
 3. How does statement nineteen describe the denominator of B_n ?
 - a. As the product of the primes that are one more than a factor of n , unity and n included
 - b. As the product of all primes below n
 - c. As n factorial
 - d. As the square of n
 4. Why is it unnecessary to write the odd factors of n , except unity?
 - a. Because odd factors never divide n
 - b. Because the next numbers to odd factors are even and hence composite
 - c. Because they give the numerator instead
 - d. Because they are always prime

Answers

1. **b)** He says that to know these properties it will be convenient to calculate a few B 's by substituting 3, 5, 7, 9, ... for n in equation twelve.
2. **b)** Statement fifteen: the denominator of B_n contains each of the factors 2 and 3 once and only once.

3. a) The denominator is the continued product of the prime numbers which are the next numbers to the factors of n , including unity and the number itself.

4. b) The next number after an odd factor is even, hence composite, so it never contributes a prime to the denominator.

Worked examples

The denominator of B_{24} Use statement nineteen to find the denominator of B_{24} .

Write all the factors of 24: 1, 2, 3, 4, 6, 8, 12, 24.

Add one to each: 2, 3, 4, 5, 7, 9, 13, 25.

Strike out the composite numbers 4, 9 and 25, leaving the primes 2, 3, 5, 7, 13.

Multiply them: $2 \times 3 \times 5 \times 7 \times 13 = 2730$.

Answer: 2730

Practice

1. Use statement nineteen to find the denominator of B_{20} .

Hint: Unity and the even factors of 20 are 1, 2, 4, 10, 20; add one to each.

Answer: 330: the next numbers are 2, 3, 5, 11, 21; striking out 21 leaves 2, 3, 5, 11, whose product is 330, as in the table.

2. From the equation $B_{10} - (5/2)B_4 = -1/132$ and $B_4 = 1/30$, find B_{10} .

Hint: Five halves of one thirtieth is one twelfth.

Answer: $B_{10} = 1/12 - 1/132 = 10/132 = 5/66$.

3. Check statement sixteen for $n = 4$: is $2^4(2^4 - 1)B_4/4$ an integer, and is $2(2^4 - 1)B_4$ odd?

Hint: Use $B_4 = 1/30$ and $2^4 - 1 = 15$.

Answer: Yes: $16 \times 15 \times (1/30) / 4 = 2$, an integer, and $2 \times 15 \times (1/30) = 1$, an odd integer.

Sections 8 to 11

Section 8.

fractional part of B_n

$$B_{16} : \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = \frac{47}{510}$$

$$B_{22} : 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$$

$$B_{28} : \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{29} - 1 = \frac{59}{870}$$

Fractional parts from primes

Again, taking the fractional part of any B and splitting it into partial fractions, we see that the fractional part of B_n equals $(-1)^{(n/2)}$ times the sum of the reciprocals of the prime factors of the denominator of B_n , minus $(-1)^{(n/2)}$; this is statement twenty.

Thus the fractional part of $B_{16} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = \frac{47}{510}$; that of $B_{22} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$; that of $B_{28} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{29} - 1 = \frac{59}{870}$; and so on.

Section 9.

It can be inferred from statement twenty that, if G be the G.C.M. and L the L.C.M. of the denominators of B_m and B_n , then L/G is the denominator of $B_m - (-1)^{((m-n)/2)} B_n$, and hence, if the denominators of B_m and B_n are equal, then $B_m - (-1)^{((m-n)/2)} B_n$ is an integer; this is statement twenty-one.

Example: $B_{24} - B_{12}$ and $B_{32} - B_{16}$ are integers, while the denominator of $B_{10} + B_{20}$ is 5.

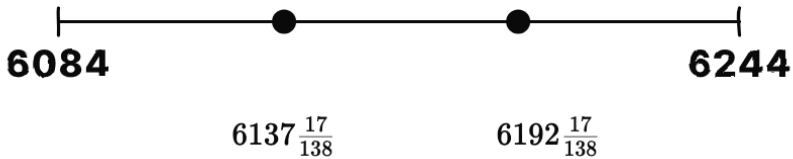
It will be observed that the following hold. First, if n is a multiple of 4, then the numerator of $B_n - 1/30$ in its lowest terms is divisible by 20; but if n is not a multiple of 4 then that of $B_n/n - 1/12$ in its lowest terms is divisible by 5; this is statement twenty-two. Second, if n is any integer, then $2(2^{4n+2} - 1)B_{4n+2}/(2n + 1)$, $2(2^{8n+4} - 1)B_{8n+4}/(2n + 1)$ and $2(2^{8n+4} - 1)B_{16n+8}/(2n + 1)$ are integers of the form $30p + 1$; this is statement twenty-three.

Section 10.

$$\frac{17}{138}$$

$$11 \mid \text{top}$$

$$5 \mid B - \frac{11}{6}$$



$$B_{22} = \frac{854513}{138}$$

Pinning down B22

If a B is known to lie between certain limits, then it is possible to find its exact value from the above properties.

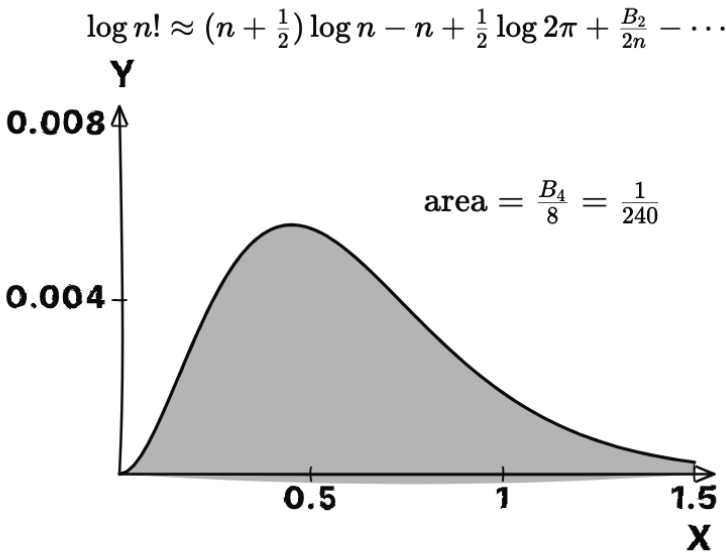
Suppose we know that B22 lies between 6084 and 6244; its exact value can be found as follows.

The fractional part of B22 is $\frac{17}{138}$ by statement twenty; also B22 is divisible by 11 by statement eighteen. And by statement twenty-two, $B_{22} - \frac{11}{6}$ must be divisible by 5. To satisfy these conditions B22 must be either $6137 \frac{17}{138}$ or $6192 \frac{17}{138}$.

But according to statement eighteen the numerator of B22 should be a prime number after it is divided by 11; and consequently B22 must be

equal to $6192 \frac{17}{138}$, or $854513/138$, since the numerator of $6137 \frac{17}{138}$ is divisible not only by 11 but also by 7 and 17.

Section 11.



An area equal to $B_4/8$

It is known, from Edwards' Differential Calculus, chapter five, example 29, that $B_n = (2 \cdot n!/(2\pi)^n)(1/1^n + 1/2^n + 1/3^n + \dots)$, or $2 \cdot n!/(2\pi)^n = B_n(1 - 1/2^n)(1 - 1/3^n)(1 - 1/5^n) \dots$, equation twenty-four, where 2, 3, 5 and so on are prime numbers.

Also $B_n/(2n) = \int_0^\infty x^{n-1}/(e^{(2\pi x)} - 1) dx$, equation twenty-five. For $\int_0^\infty x^{n-1}/(e^{(2\pi x)} - 1) dx = \int_0^\infty x^{n-1}(e^{(-2\pi x)} + e^{(-4\pi x)} + \dots) dx = ((n-1)!/(2\pi)^n)(1/1^n + 1/2^n + 1/3^n + \dots) = B_n/(2n)$, by equation twenty-four. In a similar manner $\int_0^\infty x^n/(e^{(\pi x)} - e^{(-\pi x)})^2 dx =$

$B_n/(4\pi)$, and $\int_0^\infty x^{n-2} \log(1 - e^{-2\pi x}) dx = -\pi B_n/(n(n-1))$, equation twenty-six.

Take logarithms of both sides in equation twenty-four and write for $\log_e n!$ the well-known expansion of $\log_e \Gamma(n+1)$, as in Carr's Synopsis, namely $(n + \frac{1}{2}) \log n - n + \frac{1}{2} \log 2\pi + B_2/(1 \cdot 2n) - B_4/(3 \cdot 4n^3) + B_6/(5 \cdot 6n^5) - \dots - (-1)^p B_{2p}\theta/((2p-1)2p n^{(2p-1)})$, equation twenty-seven, where θ lies between 0 and 1.

Here $B_{2p}\theta/((2p-1)2p n^{(2p-1)}) = B_{2p}/((2p-1)2p n^{(2p-1)}) - B_{2p+2}/((2p+1)(2p+2)n^{(2p+1)}) + \dots$. This equals $-(1/\pi) \int_0^\infty (x^{(2p-2)}/n^{(2p-1)}) \log(1 - e^{-2\pi x}) dx + (1/\pi) \int_0^\infty (x^{(2p)}/n^{(2p+1)}) \log(1 - e^{-2\pi x}) dx - \dots$. Collecting the terms, it equals $-(1/\pi) \int_0^\infty (x^{(2p-2)}/(n^{(2p-3)}(n^2 + x^2))) \log(1 - e^{-2\pi x}) dx$, which is $-\int_0^\infty x^{(2p-2)} \log(1 - e^{-2\pi nx}) / (\pi(1 + x^2)) dx$.

We can find the integral part of B_n , and since the fractional part can be found, as shewn in section 8, the exact value of B_n is known. Unless the calculation is made to depend upon the values of $\log_{10} e$, $\log_e 10$, π and so on, which are known to a great number of decimal places, we should have to find the logarithms of certain numbers whose values are not found in the tables to as many places of decimals as we require. Such difficulties are removed by the method given in section 13.

Questions

1. According to statement twenty, what is the fractional part of B_n made of?

- a. The reciprocals of all the factors of n
 - b. The sum of the reciprocals of the prime factors of its denominator, minus one, with the sign $(-1)^{(n/2)}$
 - c. The numerator divided by n
 - d. The square root of the denominator
2. Why is $B_{24} - B_{12}$ an integer, according to statement twenty-one?
- a. Because both are integers
 - b. Because their denominators are equal, so L/G is 1
 - c. Because 24 is twice 12
 - d. Because both numerators are prime
3. In section 10, what conditions narrow B_{22} down to two candidates?
- a. Its fractional part $17/138$, divisibility of the numerator by 11, and $B_{22} - 11/6$ divisible by 5
 - b. Its value of π and its logarithm
 - c. The table of denominators alone
 - d. The integral in equation twenty-five
4. What does equation twenty-four connect B_n with?
- a. The sum of the reciprocals of the n -th powers of the whole numbers
 - b. The number of primes below n
 - c. The derivative of $\cot x$ at zero
 - d. The digits of π

Answers

1. **b)** Splitting the fractional part into partial fractions, it equals $(-1)^{(n/2)}$ times the sum of the reciprocals of the prime factors of the denominator, minus $(-1)^{(n/2)}$.
2. **b)** If the denominators of B_m and B_n are equal, then $B_m - (-1)^{((m-n)/2)} B_n$ is an integer; B_{24} and B_{12} both have denominator 2730.
3. **a)** The fractional part is $17/138$ by statement twenty, the numerator is divisible by 11 by statement eighteen, and $B_{22} - 11/6$ must be divisible by 5 by statement twenty-two.
4. **a)** $B_n = (2 \cdot n! / (2\pi)^n) (1/1^n + 1/2^n + 1/3^n + \cdots)$, which can also be written as a product over the primes.

Worked examples

Finding B_{22} from limits Suppose B_{22} lies between 6084 and 6244. Use the fractional part $17/138$, divisibility of the numerator by 11, and the rule that $B_{22} - 11/6$ is divisible by 5 to find the candidates.

Write $B_{22} = k + 17/138$ with k a whole number between 6084 and 6243; its numerator is $138k + 17$.

Keep only the k for which $138k + 17$ is divisible by 11 and the numerator of $B_{22} - 11/6$ is divisible by 5.

Exactly two remain: $k = 6137$ and $k = 6192$.

The book then rejects $6137 \cdot 17/138$, whose numerator 846923 is divisible by 7 and 17 as well as 11, and keeps $6192 \cdot 17/138 = 854513/138$.

Answer: $B_{22} = 854513/138$, as in the table

Practice

1. Check statement twenty for $B_{16} = 3617/510$: compute $1/2 + 1/3 + 1/5 + 1/17 - 1$ and compare with the fractional part of B_{16} .

Hint: Put everything over 510.

Answer: $(255 + 170 + 102 + 30 - 510)/510 = 47/510$, and $3617/510 = 7 + 47/510$, so they agree.

2. Check statement twenty-two for $n = 12$: is the numerator of $B_{12} - 1/30$ divisible by 20?

Hint: Write $1/30$ as $91/2730$.

Answer: Yes: $691/2730 - 91/2730 = 600/2730 = 20/91$, whose numerator is 20.

3. Check the first expression of statement twenty-three for $n = 1$: is $2(2^6 - 1)B_6/3$ of the form $30p + 1$?

Hint: Use $B_6 = 1/42$ and $2^6 - 1 = 63$.

Answer: Yes: $2 \times 63 \times (1/42)/3 = 1$, which is $30 \times 0 + 1$.

Section 12

Results fourteen to seventeen, twenty and twenty-one can be obtained as follows. We have $1/(2x^2) + 1/(x + 1)^2 + 1/(x + 2)^2 + 1/(x + 3)^2 + 1/(x + 4)^2 + \cdots - 1/x - 1/(6(x^3 - x)) + 1/(5(x^5 - x)) + 1/(7(x^7 - x)) + 1/(11(x^{11} - x)) + \cdots = 1/x^{15} - 7/x^{17} + 55/x^{19} - 529/x^{21} + \cdots$, equation twenty-eight, where 5, 7, 11, 13 and so on are prime numbers above 3. If we can prove that the left-hand side of equation twenty-eight can be expanded in ascending powers of $1/x$ with integral coefficients, then statements twenty and twenty-one are at once deduced as follows.

IF WE CAN PROVE

LEFT SIDE OF (28)

WHOLE NUMBERS?

$$\frac{1}{x^{15}} - \frac{7}{x^{17}} + \frac{55}{x^{19}} - \frac{529}{x^{21}} + \cdots$$

One claim about a series

From equation twenty-seven we have $d^2 \log \Gamma(n+1)/dn^2 = 1/(n+1)^2 + 1/(n+2)^2 + 1/(n+3)^2 + \dots = 1/n - 1/(2n^2) + B_2/n^3 - B_4/n^5 + B_6/n^7 - B_8/n^9 + \dots - (-1)^p B_{2p}/n^{2p+1}$, equation twenty-nine.

$$\sum_{k \geq 1} \frac{1}{(n+k)^2} = \frac{1}{n} - \frac{1}{2n^2} + \frac{B_2}{n^3} - \frac{B_4}{n^5} + \dots$$

$$\text{rest} = \int_0^\infty \frac{\pi x^{2p} dx}{(1+x^2) \sinh^2 \pi n x}$$

MUST BE WHOLE NUMBERS; ALL B POSITIVE HERE

$$B_2 - \frac{1}{6} = \frac{1}{6} - \frac{1}{6} = 0$$

$$-B_4 - \frac{1}{6} + \frac{1}{5} = -\frac{1}{30} - \frac{1}{6} + \frac{1}{5} = 0$$

$$B_6 - \frac{1}{6} + \frac{1}{7} = \frac{1}{42} - \frac{1}{6} + \frac{1}{7} = 0$$

$$-B_8 - \frac{1}{6} + \frac{1}{5} = -\frac{1}{30} - \frac{1}{6} + \frac{1}{5} = 0$$

$$B_{10} - \frac{1}{6} + \frac{1}{11} = \frac{5}{66} - \frac{1}{6} + \frac{1}{11} = 0$$

$$-\frac{1}{6} = \frac{1}{2} + \frac{1}{3} - 1$$

The coefficients that must be whole

Here $B_{2p}/n^{2p+1} = B_{2p}/n^{2p+1} - B_{2p+2}/n^{2p+3} + \dots$. This equals $4\pi \int_0^\infty x^{2p}/(n^{2p+1}(e^{(\pi x)} - e^{(-\pi x)})^2) dx - 4\pi \int_0^\infty x^{2p+2}/(n^{2p+3}(e^{(\pi x)} - e^{(-\pi x)})^2) dx + \dots$. That is $\pi \int_0^\infty (x^{2p}/n^{2p+1} - x^{2p+2}/n^{2p+3} + \dots) dx/\sinh^2 \pi x$. And that is $\pi \int_0^\infty x^{2p}/(n^{2p-1}(n^2 + x^2)) dx/\sinh^2 \pi x = \int_0^\infty \pi x^{2p}/((1+x^2) \sinh^2 \pi n x) dx$.

Substituting the result of equation twenty-nine in equation twenty-eight we see that $B_2/x^3 - B_4/x^5 + B_6/x^7 - \dots - 1/(6(x^3 - x)) + 1/(5(x^5$

$-x)) + 1/(7(x^7 - x)) + 1/(11(x^{11} - x)) + \dots$, where 5, 7, 11 and so on are prime numbers, can be expanded in ascending powers of $1/x$ with integral coefficients.

Therefore $B_2 = -1/6$, $-B_4 = -1/6 + 1/5$, $B_6 = -1/6 + 1/7$, $-B_8 = -1/6 + 1/5$, $B_{10} = -1/6 + 1/11$, and so on, which are the coefficients of $1/x^3$, $1/x^5$, $1/x^7$ and so on, are integers.

Writing $1/2 + 1/3 - 1$ for $-1/6$ we get the results of statements twenty and twenty-one.

Again changing n to $1/2n$ in equation twenty-nine, and subtracting half of the result from equation twenty-nine, we have $1/(n+1)^2 - 1/(n+2)^2 + 1/(n+3)^2 - \dots = 1/(2n^2) - (2^2 - 1)B_2/n^3 + (2^4 - 1)B_4/n^5 - (2^6 - 1)B_6/n^7 + \dots + (-1)^p (2^{2p} - 1)B_{2p}\theta/n^{2p+1}$, equation thirty, where θ lies between 0 and 1, and also, by equation twenty-nine, $(2^{2p} - 1)B_{2p}\theta/n^{2p+1} = \int_0^\infty \pi x^{2p} \cosh \pi x / ((1+x^2) \sinh^2 \pi x) dx$.

ALTERNATING (30)



ODDNESS, PART 2

IF WE CAN PROVE

HARMONIC (31)



ODDNESS, PART 1

IF WE CAN PROVE

Two more routes, the same gap

Thus we see that, if we can prove that twice the left-hand side of equation thirty can be expanded in ascending powers of $1/n$ with integral coefficients, then the second part of statement sixteen is at once proved.

Again from equation twenty-seven we have $d \log \Gamma(n+1)/dn = 1 + 1/2 + 1/3 + 1/4 + \dots + 1/n - \gamma = \log n + 1/(2n) - B_2/(2n^2) + B_4/(4n^4) - B_6/(6n^6) + B_8/(8n^8) - \dots + (-1)^p B_{2p}\theta/(2p n^{(2p)})$, equation thirty-one, where θ lies between 0 and 1; and also, by equation twenty-five, $B_{2p}\theta/(2p n^{(2p)}) = \int_0^\infty 2x^{(2p-1)} / ((1+x^2)(e^{(2\pi nx)} - 1)) dx$, from which it can easily be shewn that $1/(n+2) - 1/(n+4) + 1/(n+6) - 1/(n+8) + 1/(n+10) - \dots = 1/(2n) - 2(2^2-1)B_2/(2n^2) + 2^3(2^4-1)B_4/(4n^4) - 2^5(2^6-1)B_6/(6n^6) + \dots + (-1)^p 2^{(2p-1)}$

$(2^{2p} - 1)B_{2p}\theta/(2p n^{2p}) - \dots$, equation thirty-two, where θ lies between 0 and 1; and also, by equation thirty-one, $2^{2p-1}(2^{2p} - 1)B_{2p}\theta/(2p n^{2p}) = \int_0^\infty x^{2p-1}/(2(1+x^2) \sinh(\frac{1}{2}\pi nx)) dx$.

From the above theorem we see that, if we can prove that $2(1/(n+2) - 1/(n+4) + 1/(n+6) - \dots)$ can be expanded in ascending powers of $1/n$ with integral coefficients, then the first part of statement sixteen at once follows.

Questions

1. On what single claim does Ramanujan make statements twenty and twenty-one depend in section 12?
 - a. That the left-hand side of equation twenty-eight can be expanded in ascending powers of $1/x$ with integral coefficients
 - b. That all Bernoulli numbers are integers
 - c. That π is irrational
 - d. That equation twelve has no solution
2. Which quantities does Ramanujan conclude are integers?
 - a. $B_2 - \frac{1}{6}$, $-B_4 - \frac{1}{6} + \frac{1}{5}$, $B_6 - \frac{1}{6} + \frac{1}{7}$, and so on, the coefficients of $1/x^3$, $1/x^5$, $1/x^7$
 - b. The Bernoulli numbers themselves
 - c. The values of θ
 - d. The integrals in equation twenty-five

3. What would prove the second part of statement sixteen, according to section 12?
 - a. That twice the left-hand side of equation thirty can be expanded in ascending powers of $1/n$ with integral coefficients
 - b. That equation twenty-four holds
 - c. That the denominator of every B is 6
 - d. That the series in equation thirty-two converges

Answers

1. a) If we can prove that the left-hand side of equation twenty-eight can be expanded in ascending powers of $1/x$ with integral coefficients, then statements twenty and twenty-one are at once deduced.

2. a) Therefore $B_2 - \frac{1}{6}$, $-B_4 - \frac{1}{6} + \frac{1}{5}$, $B_6 - \frac{1}{6} + \frac{1}{7}$, ... which are the coefficients of $1/x^3$, $1/x^5$, $1/x^7$, ... are integers.

3. a) If we can prove that twice the left-hand side of equation thirty can be expanded in ascending powers of $1/n$ with integral coefficients, then the second part of statement sixteen is at once proved.

Worked examples

The integral coefficients for small indices Check that the coefficients $-B_4 - \frac{1}{6} + \frac{1}{5}$ and $B_{14} - \frac{1}{6}$ are integers, using $B_4 = 1/30$ and $B_{14} = 7/6$.

Put $-B_4 - \frac{1}{6} + \frac{1}{5}$ over 30: $-1/30 - 5/30 + 6/30$.

The sum is $0/30 = 0$, an integer.

For $B_{14} - \frac{1}{6}$: $7/6 - 1/6 = 6/6$.

That is 1, an integer, and it matches the first term $1/x^{15}$ on the right of equation twenty-eight.

Answer: Both are integers: 0 and 1.

Practice

1. Check that $B_6 - \frac{1}{6} + \frac{1}{7}$ is an integer, with $B_6 = \frac{1}{42}$.

Hint: Put everything over 42.

Answer: $\frac{1}{42} - \frac{7}{42} + \frac{6}{42} = 0$, an integer.

2. Check that $B_{10} - \frac{1}{6} + \frac{1}{11}$ is an integer, with $B_{10} = \frac{5}{66}$.

Hint: Put everything over 66.

Answer: $\frac{5}{66} - \frac{11}{66} + \frac{6}{66} = 0$, an integer.

3. Why does writing $\frac{1}{2} + \frac{1}{3} - 1$ for $-\frac{1}{6}$ turn these integer coefficients into statement twenty?

Hint: Look at which primes appear in each coefficient.

Answer: Each coefficient is $\pm B_n$ plus a sum of reciprocals of primes: $\frac{1}{2}$ and $\frac{1}{3}$ from $-\frac{1}{6}$, and $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{11}$ and so on from the other terms. Saying this is an integer means the fractional part of B_n is $\pm$ the sum of those reciprocals minus one, which is statement twenty.

Sections 13 to 15

Section 13.

HOW MANY DIGITS?

$$\log_{10} B_n \approx \left(n + \frac{1}{2}\right) \log_{10} n - 1.2324743503 n + 0.700120$$

$$\text{true value} \approx \dots + \frac{0.0362}{n}$$

$$\log_e B_n \approx \left(n + \frac{1}{2}\right) \log_e n - n(1 + \log_e 2\pi) + \frac{1}{2} \log_e 8\pi$$

$$\times \log_{10} e = 0.4342944819$$

$$n = 20 : \quad 2.7218 + 0.0018 = 2.7236 \approx \log_{10} 529.12$$

Counting digits

The first few digits, and the number of digits in the integral part as well as in the numerator of B_n , can be found from the approximate formula $\log_{10} B_n = (n + 1/2) \log_{10} n - 1.2324743503 n + 0.700120$, the true value being greater by about $0.0362/n$ when n is great; this is result thirty-three.

This formula is proved as follows: taking logarithms of both sides in equation twenty-four, $\log_e B_n = (n + 1/2) \log_e n - n(1 + \log_e 2\pi) + 1/2$

$\log_e 8\pi$, nearly. Multiplying both sides by $\log_{10} e$, or .4342944819, and reducing, we can get the result.

Section 14.

$$\frac{B_n}{B_{n-2}} \rightarrow \frac{n(n-1)}{4\pi^2}$$

$$\pi = 3.14159\ 26535\ 89793 \dots$$

rough ratio $\Rightarrow$ between $\Rightarrow$ exact

$$B_{12} \in (0, 1) : \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} - 1 = \frac{691}{2730}$$

$$B_{14} \in (0, 2) : 2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$$

$$B_{16} \in (7, 8) : 6 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = \frac{3617}{510}$$

$$B_{18} \in (54, 55) : 56 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7} - \frac{1}{19} = \frac{43867}{798}$$

$$B_{20} \in (529, 530) : 528 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{11} = \frac{174611}{330}$$

Each B from the one before

Changing n to n minus 2 in equation twenty-four and taking the ratio of the two results, we have $B_n = (n(n-1)/(4\pi^2)) B_{n-2} (1 - (2^2 - 1)/(2^n - 1))(1 - (3^2 - 1)/(3^n - 1))(1 - (5^2 - 1)/(5^n - 1)) \dots$, equation thirty-four, where 2, 3, 5 and so on are prime numbers.

Hence we see that B_n/B_{n-2} approaches $n(n-1)/(4\pi^2)$ very rapidly as n becomes greater and greater; this is result thirty-five.

From the value of pi, namely 3.14159 26535 89793 23846 26433 83279 50288 41971 69399 and so on, the integral part of any B can be found from the previous B; and from the integral part the exact value can at once be written by help of statement twenty, as follows. The table gives, for each B, the approximate ratio to the previous B, the two whole numbers between which it lies, found by a rough calculation from the preceding B, and hence its exact value.

B₂ lies between 0 and 1, so it is $1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. B₄ = $(3 \cdot 4 / (4\pi^2))$ B₂, lies between 0 and 1, so it is $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30}$. B₆ = $(5 \cdot 6 / (4\pi^2))$ B₄, lies between 0 and 1, so it is $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7} = \frac{1}{42}$. B₈ = $(7 \cdot 8 / (4\pi^2))$ B₆, lies between 0 and 1, so it is $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1 = \frac{1}{30}$. B₁₀ = $(9 \cdot 10 / (4\pi^2))$ B₈, lies between 0 and 1, so it is $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{11} = \frac{5}{66}$. B₁₂ = $(11 \cdot 12 / (4\pi^2))$ B₁₀, lies between 0 and 1, so it is $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} - 1 = \frac{691}{2730}$. B₁₄ = $(13 \cdot 14 / (4\pi^2))$ B₁₂, lies between 0 and 2, so it is $2 - \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$. B₁₆ = $(15 \cdot 16 / (4\pi^2))$ B₁₄, lies between 7 and 8, so it is $6 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = \frac{3617}{510}$. B₁₈ = $(17 \cdot 18 / (4\pi^2))$ B₁₆, lies between 54 and 55, so it is $56 - \frac{1}{2} - \frac{1}{3} - \frac{1}{7} - \frac{1}{19} = \frac{43867}{798}$. B₂₀ = $(19 \cdot 20 / (4\pi^2))$ B₁₈, lies between 529 and 530, so it is $528 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{11} = \frac{174611}{330}$; and so on.

Section 15.

$$B_{444} = \frac{N}{D}$$

$$\text{fraction} = 23975417/90709710$$

$$N = 37 \times \text{prime}$$

$$\log_{10} B_{444} \approx 630.2433$$

WHOLE PART: 631 DIGITS, 1751...

$$\log_{10} N \approx 630.2433 + \log_{10} 90709710 = 638.2010$$

N: 639 DIGITS, 1588...

$$[B_{444}] + \frac{698392}{3023657} : 20 \mid \text{top}$$

$$[B_{444}] \text{ ends } \dots e4, e \text{ even}$$

$$N \text{ ends } \dots e57, e \text{ even}$$

B444 without writing it

From the preceding theorems we know some of the properties of B_n for all positive even values of n . As an example let us take $B_{444} = N/D$.

The fractional part of B_{444} is $23975417/90709710$ by statement twenty. The numerator of B_{444} is divisible by 37 and the quotient is a prime number by statement eighteen. Again $\log_{10} B_{444} = 630.2433$, nearly, by result thirty-three. Therefore the integral part of B_{444} contains 631 digits, the first 4 digits being 1751. Again $\log_{10} N = \log_{10} B_{444} + \log_{10} D = 630.2433 + \log_{10} 90709710 = 638.2010$, nearly. Therefore N contains 639 digits, the first four digits being 1588.

Again the numerator of $B_{444} - 1/30$ is divisible by 20; that is to say, if $[B_{444}]$ is the integral part of B_{444} , $[B_{444}] + 23975417/90709710 -$

$1/30 = [B_{444}] + 698392/3023657$ has a numerator divisible by 20. Therefore the integral part of B_{444} ends with 4 and also the figure next to the last is even.

Hence N ends with 57 and also the third figure from the last is even.

Questions

1. What can result thirty-three be used for?
 - a. To find the first few digits and the number of digits of B_n and its numerator
 - b. To find the denominator of B_n exactly
 - c. To prove statement fourteen
 - d. To compute π
2. What does result thirty-five say about the ratio B_n/B_{n-2} ?
 - a. It approaches $n(n-1)/(4\pi^2)$ very rapidly as n grows
 - b. It is always 1
 - c. It approaches π
 - d. It decreases to zero
3. How does Ramanujan get the exact value of a B once he knows the two whole numbers it lies between?
 - a. He adds the fractional part given by statement twenty
 - b. He averages the two numbers
 - c. He uses the table of denominators only
 - d. He multiplies by π

4. What does section 15 conclude about the last digits of N , the numerator of B_{444} ?
 - a. N ends with 57 and the third figure from the last is even
 - b. N ends with 00
 - c. N is a prime ending in 1
 - d. Nothing can be said

Answers

1. **a)** The first few digits, and the number of digits in the integral part and in the numerator of B_n , can be found from the approximate formula for $\log_{10} B_n$.
2. **a)** From equation thirty-four, B_n/B_{n-2} approaches $n(n-1)/(4\pi^2)$ very rapidly as n becomes greater and greater.
3. **a)** From the integral part, the exact value can at once be written by help of statement twenty, which gives the fractional part.
4. **a)** Hence N ends with 57 and also the third figure from the last is even.

Worked examples

B₁₆ from B₁₄ Using the ratio $(15 \cdot 16 / (4\pi^2))$ and $B_{14} = 7/6$, find the two whole numbers B_{16} lies between, and then its exact value with the primes 2, 3, 5, 17 of its denominator.

$4\pi^2$ is about 39.478, so $15 \times 16 / (4\pi^2) = 240 / 39.478$, about 6.079.

Multiplying by $B_{14} = 7/6$, about 1.1667, gives about 7.09, so B_{16} lies between 7 and 8.

Half of 16 is even, so the fractional part is $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = 47/510$.

So $B_{16} = 7 + 47/510 = 6 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = 3617/510$.

Answer: $B_{16} = 3617/510$

Practice

1. Use the ratio $17 \cdot 18 / (4\pi^2)$ and $B_{16} = 3617/510$ (about 7.092) to find the whole numbers between which B_{18} lies.

Hint: $17 \times 18 = 306$ and $4\pi^2$ is about 39.478.

Answer: $306/39.478$ is about 7.751, and 7.751×7.092 is about 54.97, so B_{18} lies between 54 and 55, as in the table.

2. Use result thirty-three to estimate $\log_{10} B_{20}$, and compare with the true value $\log_{10}(174611/330) = 2.72356$.

Hint: Compute $20.5 \times \log_{10} 20 - 1.2324743503 \times 20 + 0.700120$, with $\log_{10} 20 = 1.30103$.

Answer: $26.67112 - 24.64949 + 0.70012 = 2.72175$; the true value is larger by about 0.0018, close to $0.0362/20 = 0.0018$, as result thirty-three says.

Sections 16 and 17

Section 16.



$$\begin{aligned} \frac{4}{3} B_n(2^n - 1) &= c_6 B_{n-6}(2^{n-6} - 1) - c_{12} B_{n-12}(2^{n-12} - 1) + \dots \\ &+ \frac{1}{3} n(-1)^{\frac{n-2}{6}} \text{ or } \frac{1}{6} n(-1)^{\frac{n-4}{6}} \text{ or } \frac{1}{3} n(-1)^{\frac{n-6}{6}} \end{aligned}$$

(37)

$$\begin{aligned} c_3 \left(1 - \frac{1}{2^{n-4}}\right) B_{n-3} - c_9 \left(1 - \frac{1}{2^{n-10}}\right) B_{n-9} + c_{15} \left(1 - \frac{1}{2^{n-16}}\right) B_{n-15} - \dots \\ = \frac{2}{3} \cdot \frac{n}{2^n} \left(3^{\frac{n-1}{2}} + (-1)^{\frac{n-3}{6}} \text{ or } (-1)^{\frac{n+1}{6}} \text{ or } (-1)^{\frac{n+5}{6}}\right) \end{aligned}$$

Starting from tan and cosec

Instead of starting with cot x as in sections 2 and 3, we may start with tan x or cosec x and get other similar results.

Thus, first, $(4/3)B_n(2^n - 1) = c_6 B_{n-6}(2^{n-6} - 1) - c_{12} B_{n-12}(2^{n-12} - 1) + \dots + \frac{1}{3} n(-1)^{((n-2)/6)} \text{ or } \frac{1}{6} n(-1)^{((n-4)/6)} \text{ or } \frac{1}{3} n(-1)^{((n-6)/6)}$, equation thirty-six.

Second, $c_3(1 - 1/2^{n-4})B_{n-3} - c_9(1 - 1/2^{n-10})B_{n-9} + c_{15}(1 - 1/2^{n-16})B_{n-15} - \dots = (2/3)(n/2^n)(3^{((n-1)/2)} + (-1)^{((n-3)/6)} \text{ or } (-1)^{((n+1)/6)} \text{ or } (-1)^{((n+5)/6)}))$, equation thirty-seven.

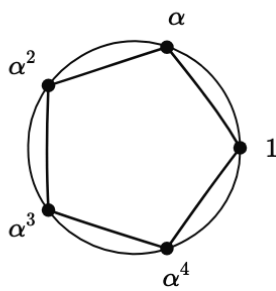
Section 17.

1 INTERVAL

2 INTERVAL

3 INTERVAL

$$\omega^p = 1 \text{ or } \omega^p = i$$



$$\begin{aligned} \varphi(x) &= \sum_{k=0}^4 \sin x \alpha^k \\ &= 5 \left(\frac{x^5}{5!} - \frac{x^{15}}{15!} + \frac{x^{25}}{25!} - \dots \right) \end{aligned}$$

$$\begin{aligned} &16 \prod_{k=0}^4 \sin x \alpha^k \\ &= \varphi(2x) + \varphi(x(1 + \sqrt{5})) + \varphi(x(1 - \sqrt{5})) \end{aligned}$$

Five roots of unity

The formulae obtained in sections 1, 3 and 4 may be called the one interval, two interval and three interval formulae respectively. The p interval formulae can be got by taking the p-th roots of unity or of i according as p is odd or even.

For example, let us take the fifth roots of unity, 1, α , α^2 , α^3 , α^4 , and find the 5 interval formulae.

Let $\varphi(x) = \sin x + \sin x\alpha + \sin x\alpha^2 + \sin x\alpha^3 + \sin x\alpha^4 = 5(x^5/5! - x^{15}/15! + x^{25}/25! - \dots)$.

Then it can easily be shewn that $16 \sin x \sin x\alpha \sin x\alpha^2 \sin x\alpha^3 \sin x\alpha^4 = \varphi(2x) - \varphi(2x(\alpha + \alpha^4)) - \varphi(2x(\alpha^2 + \alpha^3)) = \varphi(2x) + \varphi(x(1 + \sqrt{5})) + \varphi(x(1 - \sqrt{5}))$.

Taking logarithms and differentiating both sides, we have $5(B_0 + B_{10}x^{10}/10! + B_{20}x^{20}/20! + B_{30}x^{30}/30! + \dots) = -x((x^4/4!)(1 + \alpha_5) - (x^{14}/14!)(1 + \alpha_{15}) + \dots) / ((x^5/5!)(1 + \alpha_5) - (x^{15}/15!)(1 + \alpha_{15}) + \dots)$, equation thirty-eight, where $\alpha_n = ((1 + \sqrt{5})/2)^n + ((1 - \sqrt{5})/2)^n$, so that $\alpha_n\alpha_m = \alpha_{n+m} + (-1)^n\alpha_{m-n}$.

$$\begin{array}{ccccccc} \odot & \odot & \odot & \bullet & \bullet & \odot & \odot & \odot & \bullet & \bullet & \odot & \odot & \odot & \bullet & \bullet \\ & & B_4 & & & & B_{14} & & & & & & B_{24} & & \end{array}$$

$$5\left(B_0 + \frac{B_{10}x^{10}}{10!} + \frac{B_{20}x^{20}}{20!} + \dots\right) \quad (38)$$

$$= -x \frac{\frac{x^4}{4!}(1 + \alpha_5) - \frac{x^{14}}{14!}(1 + \alpha_{15}) + \dots}{\frac{x^5}{5!}(1 + \alpha_5) - \frac{x^{15}}{15!}(1 + \alpha_{15}) + \dots}$$

$$\alpha_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

$$5\left(\frac{B_2x^2}{2!} + \frac{B_{12}x^{12}}{12!} + \dots\right) = x \frac{\frac{x^6}{6!}(1 + \alpha_7) - \dots}{\frac{x^5}{5!}(1 + \alpha_5) - \dots} \quad (39)$$

$$5\left(\frac{B_4x^4}{4!} + \frac{B_{14}x^{14}}{14!} + \dots\right) = x \frac{\frac{x^8}{8!}(\alpha_7 - 1) - \dots}{\frac{x^5}{5!}(1 + \alpha_5) - \dots} \quad (40)$$

One strand per formula

Similarly, first, $5(B_2x^2/2! + B_{12}x^{12}/12! + B_{22}x^{22}/22! + \dots) = x$
 $((x^6/6!)(1 + \alpha_7) - (x^{16}/16!)(1 + \alpha_{17}) + \dots) / ((x^5/5!)(1 + \alpha_5) -$
 $(x^{15}/15!)(1 + \alpha_{15}) + \dots)$, equation thirty-nine.

Second, $5(B_4x^4/4! + B_{14}x^{14}/14! + B_{24}x^{24}/24! + \dots) = x$ $((x^8/8!)(\alpha_7 -$
 $1) - (x^{18}/18!)(\alpha_{17} - 1) + \dots) / ((x^5/5!)(1 + \alpha_5) - (x^{15}/15!)(1 + \alpha_{15}) +$
 $\dots)$, equation forty.

Third, $10(B_6x^6/6! + B_{16}x^{16}/16! + B_{26}x^{26}/26! + \dots) = x$ $((x^{10}/10!)$
 $(\alpha_{10} - 3) - (x^{20}/20!)(\alpha_{20} - 3) + (x^{30}/30!)(\alpha_{30} - 3) - \dots) / ((x^5/5!)(1$
 $+ \alpha_5) - (x^{15}/15!)(1 + \alpha_{15}) + (x^{25}/25!)(1 + \alpha_{25}) - \dots)$, equation forty-
one.



$$10\left(\frac{B_6x^6}{6!} + \frac{B_{16}x^{16}}{16!} + \dots\right) = x \frac{\frac{x^{10}}{10!}(\alpha_{10} - 3) - \dots}{\frac{x^5}{5!}(1 + \alpha_5) - \dots} \quad (41)$$

$$5\left(\frac{B_8x^8}{8!} + \frac{B_{18}x^{18}}{18!} + \dots\right) = x \frac{\frac{x^{12}}{12!}(\alpha_{11} - 1) - \dots}{\frac{x^5}{5!}(1 + \alpha_5) - \dots} \quad (42)$$

FIVE STRANDS

$$\psi(x) = \sum_{k=0}^4 \cos x \alpha^k = 5\left(1 - \frac{x^{10}}{10!} + \frac{x^{20}}{20!} - \dots\right)$$

MANY MORE IDENTITIES

The other strands, and cosines

Fourth, $5(B_8x^8/8! + B_{18}x^{18}/18! + B_{28}x^{28}/28! + \dots) = x((x^{12}/12!)(\alpha_{11} - 1) - (x^{22}/22!)(\alpha_{21} - 1) + (x^{32}/32!)(\alpha_{31} - 1) - \dots) / ((x^5/5!)(1 + \alpha_5) - (x^{15}/15!)(1 + \alpha_{15}) + (x^{25}/25!)(1 + \alpha_{25}) - \dots)$, equation forty-two.

Again from $16 \cos x \cos x\alpha \cos x\alpha^2 \cos x\alpha^3 \cos x\alpha^4 = 1 + \psi(2x) + \psi(2x(\alpha + \alpha^4)) + \psi(2x(\alpha^2 + \alpha^3))$, where $\psi(x) = \cos x + \cos x\alpha + \cos x\alpha^2 + \cos x\alpha^3 + \cos x\alpha^4 = 5(1 - x^{10}/10! + x^{20}/20! - x^{30}/30! + \dots)$, and similar relations, we can get many other identities.

Questions

1. What does section 16 say can replace the cotangent as a starting point?
 - a. The tangent or the cosecant
 - b. The logarithm
 - c. The exponential only
 - d. The square root
2. How are the p interval formulae obtained, according to section 17?
 - a. By taking the p-th roots of unity or of i, according as p is odd or even
 - b. By squaring the one interval formula p times
 - c. By dividing by p
 - d. By using p different primes

3. Which Bernoulli numbers appear on the left of equation thirty-eight?
- $B_0, B_{10}, B_{20}, B_{30}, \dots$, every tenth one
 - All of them
 - Only odd-index ones
 - B_2, B_4, B_6 only

Answers

1. **a)** Instead of starting with $\cot x$ as in sections 2 and 3, we may start with $\tan x$ or $\operatorname{cosec} x$ and get other similar results.
2. **a)** The p interval formulae can be got by taking the p -th roots of unity or of i according as p is odd or even.
3. **a)** Equation thirty-eight has $5(B_0 + B_{10}x^{10}/10! + B_{20}x^{20}/20! + B_{30}x^{30}/30! + \dots)$ on the left.

Worked examples

Equation thirty-six for $n = 6$ Check equation thirty-six for $n = 6$, where the right side reduces to $c_6 B_0(2^0 - 1)$ plus the last term $\frac{1}{3}n(-1)^{(n-6)/6}$. Use $B_6 = 1/42$.

The left side is $(4/3) \times (1/42) \times (2^6 - 1) = (4/3) \times 63/42$.

$63/42 = 3/2$, so the left side is $(4/3) \times (3/2) = 2$.

On the right, $c_6 B_0(2^0 - 1) = 0$ because $2^0 - 1 = 0$.

The last term is $\frac{1}{3} \times 6 \times (-1)^0 = 2$, so both sides equal 2.

Answer: Both sides equal 2

Practice

1. The numbers $\alpha_n = ((1 + \sqrt{5})/2)^n + ((1 - \sqrt{5})/2)^n$ are whole numbers. Find α_1 , α_2 and α_5 .

Hint: Use $\alpha_0 = 2$, $\alpha_1 = 1$ and $\alpha_{n+1} = \alpha_n + \alpha_{n-1}$.

Answer: $\alpha_1 = 1$, $\alpha_2 = 3$, and continuing 4, 7, 11, so $\alpha_5 = 11$.

2. Check the leading term of equation thirty-eight: for small x , what does $-x ((x^4/4!)(1 + \alpha_5)) / ((x^5/5!)(1 + \alpha_5))$ equal, and what does that say about B_0 ?

Hint: The factors $1 + \alpha_5$ cancel.

Answer: It is $-x (x^4/24)/(x^5/120) = -5$, and the left side starts with $5B_0$, so $B_0 = -1$.

3. Check the leading term of equation thirty-nine with $\alpha_5 = 11$ and $\alpha_7 = 29$.

Hint: Compare $x (x^6/720)(30) / ((x^5/120)(12))$ with $5B_2x^2/2$.

Answer: The right side is $x^2 \times (30/720)/(12/120) = 5x^2/12$, and $5B_2x^2/2 = 5x^2/12$ gives $B_2 = 1/6$.

Section 18 and the Editors' Note

Section 18.

FOUR INTERVAL

$$a_n = \left(1 + \frac{1}{\sqrt{2}}\right)^n + \left(1 - \frac{1}{\sqrt{2}}\right)^n$$

$$b_n = \left(1 + \frac{1}{\sqrt{2}}\right)^n - \left(1 - \frac{1}{\sqrt{2}}\right)^n$$

$$a_{m+n} = a_m a_n - a_{m-n}/2^n$$

$$b_{m+n} = a_m b_n + b_{m-n}/2^n$$

$$4\left(B_0 + \frac{B_8 x^8}{8!} + \frac{B_{16} x^{16}}{16!} + \frac{B_{24} x^{24}}{24!} + \cdots\right) \quad (43)$$

$$= -x \frac{\frac{x^3}{3!} a_2 - \frac{x^{11}}{11!} a_6 + \frac{x^{19}}{19!} a_{10} - \cdots}{\frac{x^4}{4!} a_2 - \frac{x^{12}}{12!} a_6 + \frac{x^{20}}{20!} a_{10} - \cdots}$$

Helpers built on root two

The four interval formulae can be got from the following identities.

If $a_n = (1 + 1/\sqrt{2})^n + (1 - 1/\sqrt{2})^n$ and $b_n = (1 + 1/\sqrt{2})^n - (1 - 1/\sqrt{2})^n$, so that $a_{m+n} = a_m a_n - a_{m-n}/2^n$ and $b_{m+n} = a_m b_n + b_{m-n}/2^n$, then the following hold.

First, $4(B_0 + B_8 x^8/8! + B_{16} x^{16}/16! + B_{24} x^{24}/24! + \cdots) = -x$
 $((x^3/3!)a_2 - (x^{11}/11!)a_6 + (x^{19}/19!)a_{10} - \cdots) / ((x^4/4!)a_2 -$

$(x^{12}/12!)a_6 + (x^{20}/20!)a_{10} - \dots$), equation forty-three.

Second, $4(B_2x^2/2! + B_{10}x^{10}/10! + B_{18}x^{18}/18! + \dots) = -x ((x^5/5!)a_3 - (x^{13}/13!)a_7 + (x^{21}/21!)a_{11} - \dots) / ((x^4/4!)a_2 - (x^{12}/12!)a_6 + (x^{20}/20!)a_{10} - \dots)$, equation forty-four.



$$4\left(\frac{B_2x^2}{2!} + \frac{B_{10}x^{10}}{10!} + \dots\right) = -x \frac{\frac{x^5}{5!}a_3 - \frac{x^{13}}{13!}a_7 + \dots}{\frac{x^4}{4!}a_2 - \frac{x^{12}}{12!}a_6 + \dots} \quad (44)$$

$$4\sqrt{2}\left(\frac{B_4x^4}{4!} + \frac{B_{12}x^{12}}{12!} + \dots\right) = x \frac{\frac{x^7}{7!}b_3 - \frac{x^{15}}{15!}b_7 + \dots}{\frac{x^4}{4!}a_2 - \frac{x^{12}}{12!}a_6 + \dots} \quad (45)$$

$$4\sqrt{2}\left(\frac{B_6x^6}{6!} + \frac{B_{14}x^{14}}{14!} + \dots\right) = x \frac{\frac{x^9}{9!}b_4 - \frac{x^{17}}{17!}b_8 + \dots}{\frac{x^4}{4!}a_2 - \frac{x^{12}}{12!}a_6 + \dots} \quad (46)$$

FOUR STRANDS

Four strands, spaced eight

Third, $4\sqrt{2}(B_4x^4/4! + B_{12}x^{12}/12! + B_{20}x^{20}/20! + \dots) = x ((x^7/7!)b_3 - (x^{15}/15!)b_7 + (x^{23}/23!)b_{11} - \dots) / ((x^4/4!)a_2 - (x^{12}/12!)a_6 + (x^{20}/20!)a_{10} - \dots)$, equation forty-five.

Fourth, $4\sqrt{2}(B_6x^6/6! + B_{14}x^{14}/14! + B_{22}x^{22}/22! + \dots) = x ((x^9/9!)b_4 - (x^{17}/17!)b_8 + (x^{25}/25!)b_{12} - \dots) / ((x^4/4!)a_2 - (x^{12}/12!)a_6 + (x^{20}/20!)a_{10} - \dots)$, equation forty-six.

Editors' note, from Appendix I, the notes on paper 1.

EDITORS' NOTES

p.1: B_{2n} for the usual B_n

p.1: $c_k = \frac{n!}{k!(n-k)!}$

p.4: (12) true for odd $n > 1$

✕ p.8: after (27): series diverges for $x \geq n$

✕ p.9: after (29): the same

p.9: coefficients, if integral, are odd

p.10: $\psi(z+1) - \psi(\frac{1}{2}z+1) = \log 2 - \sum \frac{(-1)^{m-1}}{z+m}$

p.11: $B_{444} = N/D$

The editors' notes

Page 1, equation one: Ramanujan writes B_{2n} where B_n is usual. Page 1, equation two: $c_k = n!/(k!(n-k)!)$, the binomial coefficient. Page 4, equation twelve: equation twelve is true if n is odd and greater than 1, and the last terms are given in the three cases where n leaves the remainder 1, minus 1 or 3 on division by 6. Page 8, the derivation after equation twenty-seven in section 11: the proof is invalid as it stands, since the series in the integrand is divergent when x is at least n . The same remark applies to page 9, the derivation after equation twenty-nine. Page 9, second line from below, section 12: the coefficients, if integral, are necessarily odd. Page 10, equation thirty-two: apply equation thirty-one to the well-known formula $\psi(z+1) - \psi(\frac{1}{2}z+1) =$

$\log 2 - \sum (-1)^{(m-1)}/(z + m)$, where $\psi(z)$ is the derivative of $\log \Gamma(z)$.
Page 11, second line from below, section 15: N and D are the numerator and denominator of B₄₄₄.

The paper is interesting as a specimen of Ramanujan's earliest manner, but the principal results are well known and the proofs are incomplete. Thus statement twenty of section 8 is simply the theorem of von Staudt and Clausen, and statement sixteen, which is a consequence of statement twenty, was known to Euler: see P. Bachmann, *Niedere Zahlentheorie*, volume two, pages 43 to 51 and page 54; or N. Nielsen, *Traité élémentaire des nombres de Bernoulli*, pages 244 to 245 and 258. Ramanujan states eight theorems, namely statements fourteen to twenty-one, embodying arithmetical properties of the B's. Of these, proofs are indicated for three, namely sixteen, twenty and twenty-one; but the theorems on which these proofs would depend, namely equation twenty-eight and the corresponding propositions about the series thirty and thirty-two, are never proved. Two other theorems, seventeen and eighteen, are stated to be corollaries of sixteen and fourteen; and fourteen, fifteen and nineteen are stated merely as conjectures.

RESULTS KNOWN, PROOFS INCOMPLETE

20: VON STAUDT–CLAUSEN

16: EULER

PROOF SKETCH	COROLLARY	CONJECTURE
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16, 20, 21	?	
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	17, 18	
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		14, 15, 19
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rest on (28), (30), (32)

NEVER PROVED

Eight statements sorted

Questions

- Which Bernoulli numbers appear on the left of equation forty-three?
 - $B_0, B_8, B_{16}, B_{24}, \dots$, spaced eight apart
 - All even-index Bernoulli numbers
 - Only B_2 and B_{10}
 - Every fifth Bernoulli number

2. What do the editors say about the proof in section 11 after equation twenty-seven?
 - a. It is invalid as it stands, since the series in the integrand diverges when x is at least n
 - b. It is complete and correct
 - c. It was copied from Euler
 - d. It proves statement fourteen
3. What is the editors' overall verdict on the paper?
 - a. Interesting as a specimen of Ramanujan's earliest manner, but its principal results are well known and its proofs incomplete
 - b. A complete and original solution of all its problems
 - c. Entirely wrong
 - d. Of purely historical interest, with no correct results
4. Which statements do the editors say were stated merely as conjectures?
 - a. Fourteen, fifteen and nineteen
 - b. Sixteen, twenty and twenty-one
 - c. Seventeen and eighteen
 - d. All eight

Answers

1. **a)** Equation forty-three has $4(B_0 + B_8x^8/8! + B_{16}x^{16}/16! + B_{24}x^{24}/24! + \dots)$ on the left.
2. **a)** The proof is invalid as it stands, since the series in the integrand is divergent when x is at least n .

3. a) The paper is interesting as a specimen of Ramanujan's earliest manner, but the principal results are well known and the proofs are incomplete.

4. a) Fourteen, fifteen and nineteen are stated merely as conjectures; seventeen and eighteen are corollaries, and proofs are indicated for sixteen, twenty and twenty-one.

Practice

1. Compute $a_2 = (1 + 1/\sqrt{2})^2 + (1 - 1/\sqrt{2})^2$.

Hint: Expand both squares; the middle terms cancel.

Answer: $a_2 = 2(1 + 1/2) = 3$, since $(1 \pm 1/\sqrt{2})^2 = 1 \pm \sqrt{2} + 1/2$.

2. Check the leading term of equation forty-three: for small x , what does $-x((x^3/3!)a_2)/((x^4/4!)a_2)$ equal, and what does it say about B_0 ?

Hint: The factors a_2 cancel.

Answer: It is $-x(x^3/6)/(x^4/24) = -4$, and the left side starts with $4B_0$, so $B_0 = -1$.

3. According to the editors, on which unproved results do Ramanujan's indicated proofs of statements sixteen, twenty and twenty-one depend?

Hint: Think of the series of section 12.

Answer: On equation twenty-eight and the corresponding propositions about the series thirty and thirty-two, which are never proved.