

Collected Papers of Srinivasa Ramanujan



Srinivasa Ramanujan · 1927

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Numbers Hidden in the Cotangent

Take a calculator, set it to radians, and work out $0.1 \times \cot 0.1$: you get 0.9966644. Now compute $1 - 0.01/3 - 0.0001/45$ by hand, and the same digits appear. Those small fractions, one third and one forty-fifth, are made of a famous family of numbers, and in 1911 a young clerk in Madras, Ramanujan, published his first paper about them.

$$\begin{array}{ccc} 0.1 \cot 0.1 & & 1 - \frac{0.01}{3} - \frac{0.0001}{45} \\ \boxed{0.9966644} & = & 0.9966644 \end{array}$$

$$x \cot x = 1 - \frac{B_2}{2!}(2x)^2 - \frac{B_4}{4!}(2x)^4 - \dots$$

$$B_2 = \frac{1}{6} \Rightarrow \frac{x^2}{3}$$

$$B_4 = \frac{1}{30} \Rightarrow \frac{x^4}{45}$$

$$B_0 = -1$$

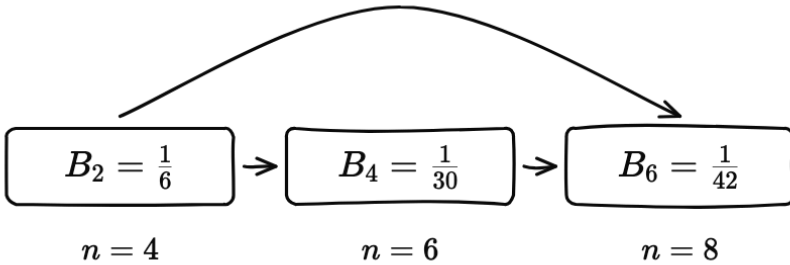
Two roads to 0.9966644

He starts from the expansion $x \cot x = 1 - (B_2/2!)(2x)^2 - (B_4/4!)(2x)^4 - (B_6/6!)(2x)^6 - \dots$, his equation one. In words: the function x times cotangent x is a sum of even powers of x , and each coefficient carries a

number B_n , a Bernoulli number in Ramanujan's convention, always positive here. Where it comes from: the cotangent is cosine over sine, both are known power series, and dividing one series by the other gives this one. With $B_2 = 1/6$ the second term is $(1/6)/2 \times 4x^2 = x^2/3$, and with $B_4 = 1/30$ the third is $x^4/45$: exactly the fractions on your calculator. Reading the constant term the same way, he lets $B_0 = -1$, so that the first term fits the pattern too.

Next he writes the cotangent three ways: as $\cos x / \sin x$, as $\sin 2x / (1 - \cos 2x)$, and as $(1 + \cos 2x) / \sin 2x$, each with its top and bottom expanded as power series. Multiply each equation by its denominator and both sides become plain series; two series are equal only if every power of x has the same coefficient, so one comparison gives one equation between the B 's. The even case is $c_2 B_{n-2} - c_4 B_{n-4} + c_6 B_{n-6} - \dots + (-1)^{((n-2)/2)} B_0 + (n/2)(-1)^{((n-2)/2)} = 0$, equation three, where c_k means the number of ways to choose k things out of n . Try n equal to 4: $6 \times B_2 + 1 - 2 = 0$, so $B_2 = 1/6$. Try n equal to 6: $15B_4 - 15 \times 1/6 - 1 + 3 = 0$, so $B_4 = 1/30$. Equations two and four do the same for odd n , with powers of two in the first.

ALL EARLIER



$$(x \cot x)^2 = -x^2 \left(1 + \frac{d \cot x}{dx} \right)$$

$$\frac{1}{2}(n+1)B_n = c_2 B_2 B_{n-2} + \cdots$$

Solving equation three step by step

Try it yourself

With pencil and paper, put n equal to 8 into equation three and solve for B_6 . You should find $28B_6 - 70/30 + 28/6 + 1 - 4 = 0$, and so $B_6 = 1/42$.

The trouble, Ramanujan says, is that every new B needs all the earlier ones, about half of n terms each time, and the work grows tedious. His first shortcut squares the series: $(x \cot x)^2 = -x^2(1 + d \cot x / dx)$, an identity anyone can check from the derivative of the cotangent.

Comparing coefficients gives $\frac{1}{2}(n+1)B_n = c_2 B_2 B_{n-2} + c_4 B_4 B_{n-4} + \cdots$, result five, which stops at the middle with a square term when half of

n is even. For n equal to 4 it reads $(5/2)B_4 = 1/2 \times 6 \times (1/6)^2 = 1/12$, again $B_4 = 1/30$. The same trick works with $d \tan x / dx = 1 + \tan^2 x$.

Remember

Ramanujan defines his Bernoulli numbers as the coefficients in the series for $x \cot x$, all positive, with B_0 taken as minus one. Writing the cotangent as a ratio of series and matching each power of x turns that definition into equations that produce the B 's one after another: one sixth, one thirtieth, one forty-second. Each such equation uses all of the earlier B 's, so the work grows quickly. Squaring $x \cot x$ gives a second relation that uses products of B 's and reaches the same values, the first hint that cleverer identities can shorten the work.

Questions

1. Why are the fractions one third and one forty-fifth in the calculator check not accidents?
 - a. They are rounding errors of the calculator
 - b. They are the coefficients of $x \cot x$, built from $B_2 = 1/6$ and $B_4 = 1/30$ by equation (1)
 - c. They are the first two prime denominators
 - d. They come from the sine series alone
2. Why does Ramanujan set $B_0 = -1$?

- a. So that the constant term of $x \cot x$ fits the same pattern as the other terms
 - b. Because all Bernoulli numbers are negative
 - c. So that equation three has no solution for odd n
 - d. Because $\cot 0$ equals -1
3. What makes comparing coefficients a legitimate way to get equations?
- a. Two series can be equal only if every power of x has the same coefficient on both sides
 - b. Every power series has the same coefficients
 - c. The coefficients are always integers
 - d. Coefficients can be chosen freely
4. Which identity gives result five, the relation with products of B's?
- a. $\sin^2 x + \cos^2 x = 1$
 - b. $(x \cot x)^2 = -x^2(1 + d \cot x / dx)$
 - c. $\cot x = \cos x / \sin x$
 - d. $d \sin x / dx = \cos x$

Answers

- 1. b)** With $B_2 = 1/6$ the second term of equation (1) is $x^2/3$, and with $B_4 = 1/30$ the third is $x^4/45$: the fractions on the calculator.
- 2. a)** Reading the constant term the same way as the others, he lets B_0 be -1 so that the first term fits the pattern.
- 3. a)** After multiplying by the denominator both sides are plain series, and two series are equal only if every power of x has the same coefficient.

4. b) His first shortcut squares the series using $(x \cot x)^2 = -x^2(1 + d \cot x / dx)$ and compares coefficients.

Worked examples

B₂ and B₄ from equation three With $B_0 = -1$, use equation three for $n = 4$ and $n = 6$ to find B_2 and B_4 ; c_k is the number of ways to choose k things out of n .

For $n = 4$, $c_2 = 6$ and the sign $(-1)^{((4-2)/2)}$ is -1 , so the equation is $6B_2 - B_0 - 2 = 0$.

Putting $B_0 = -1$ gives $6B_2 + 1 - 2 = 0$, so $B_2 = 1/6$.

For $n = 6$, $c_2 = c_4 = 15$ and the sign is $+1$, so $15B_4 - 15B_2 + B_0 + 3 = 0$.

That is $15B_4 - 15/6 - 1 + 3 = 0$, so $15B_4 = 1/2$.

Answer: $B_2 = 1/6$, $B_4 = 1/30$

Practice

1. Put $n = 8$ into equation three and find B_6 .

Hint: $c_2 = 28$, $c_4 = 70$, $c_6 = 28$, and the sign $(-1)^{((8-2)/2)}$ is -1 , so the B_0 term becomes $+1$.

Answer: $B_6 = 1/42$: $28B_6 - 70/30 + 28/6 + 1 - 4 = 0$ gives $28B_6 = 2/3$.

2. Use result five for $n = 4$ to get B_4 from $B_2 = 1/6$.

Hint: Half of 4 is even, so the only term is $1/2 c_2 (B_2)^2$ with $c_2 = 6$.

Answer: $B_4 = 1/30$: $(5/2)B_4 = 1/2 \times 6 \times 1/36 = 1/12$.

3. Using $B_6 = 1/42$ in equation (1), what is the coefficient of x^6 in $x \cot x$?

Hint: The term is $(B_6/6!)(2x)^6$, and $6! = 720$, $2^6 = 64$.

Answer: $-2/945$: $(1/42)/720 \times 64 = 64/30240 = 2/945$, with a minus sign from equation (1).

Making Half the Terms Vanish

Stand two people on a seesaw who weigh exactly the same, one at each end: the plank stays level, because what one pushes down the other pushes up. Ramanujan's next move works like that seesaw, with functions instead of people. He adds two series whose terms are equal in some places and opposite in others, and every opposite pair cancels.



COT	B_0	$+B_2$	$+B_4$	$+B_6$	$+B_8$	$+B_{10}$
COTH	B_0	$-B_2$	$+B_4$	$-B_6$	$+B_8$	$-B_{10}$
SUM	$2B_0$	0	$2B_4$	0	$2B_8$	0

Twins that cancel

The partner of the cotangent is the hyperbolic cotangent, coth, which is the cotangent with an imaginary argument: $\coth \frac{1}{2}x = i \cot \frac{1}{2}ix$,

where i is the square root of minus one. Its series has the same coefficients as the cotangent's, with every second sign flipped. So he writes $-\frac{1}{2}x(\cot \frac{1}{2}x + \coth \frac{1}{2}x) = 2(B_0 + B_4x^4/4! + B_8x^8/8! + \dots)$. In words: adding the two keeps only the terms whose index is a multiple of four, B_0 , B_4 , B_8 , and doubles them; the B_2 , B_6 , B_{10} terms cancel like the seesaw.

To turn the left side into something he can compute with, he rewrites it as one fraction and uses De Moivre's theorem, the rule for raising complex numbers to powers. The result is equation six: $2(B_0 + B_4x^4/4! + \dots) = -x(x/1! - x^5/(2^2 \cdot 5!) + x^9/(2^4 \cdot 9!) - \dots) / (x^2/2! - x^6/(2^2 \cdot 6!) + x^{10}/(2^4 \cdot 10!) - \dots)$. Check the first term: for tiny x the right side is $-x \cdot x / (x^2/2) = -2$, and the left side starts with $2B_0 = -2$, so it fits.

Subtracting instead of adding keeps the other half: $-\frac{1}{2}x(\cot \frac{1}{2}x - \coth \frac{1}{2}x) = 2(B_2x^2/2! + B_6x^6/6! + B_{10}x^{10}/10! + \dots)$, and equation seven gives it as $x(x^3/(2 \cdot 3!) - x^7/(2^3 \cdot 7!) + \dots) / (x^2/2! - x^6/(2^2 \cdot 6!) + \dots)$. For tiny x the right side is $x \cdot (x^3/12) / (x^2/2) = x^2/6$, and the left side starts with B_2x^2 , so $B_2 = 1/6$ drops out at once.

Matching coefficients as before gives equation eight, for even n greater than 2: $c_2B_{n-2}/2 - c_6B_{n-6}/2^3 + c_{10}B_{n-10}/2^5 - \dots + (n/2^{((n+2)/2)})(-1)^{(n/4)} = 0$, the sign being $(-1)^{(n/4)}$ when n is a multiple of 4 and $(-1)^{((n-2)/4)}$ when n minus 2 is. Try n equal to 6: $15 \times (1/30)/2 - (-1)/8 - 6/16 = 0.25 + 0.125 - 0.375 = 0$. The count is the point: equations two to four had $\frac{1}{2}n$ terms, all the earlier B 's; equation eight has about $\frac{1}{4}n$, only every second one. The same works with $\tan \frac{1}{2}x \pm \tanh \frac{1}{2}x$.

$$c_2 \frac{B_{n-2}}{2} - c_6 \frac{B_{n-6}}{2^3} + c_{10} \frac{B_{n-10}}{2^5} - \dots = 0$$

$$n = 6 : \quad 0.25 + 0.125 - 0.375 = 0$$

TAN ± TANH: THE SAME

EQS 2-4 $\frac{n}{2}$

EQ. 8 $\frac{n}{4}$

$$n = 4 : \quad 0.5 - 0.5 = 0$$

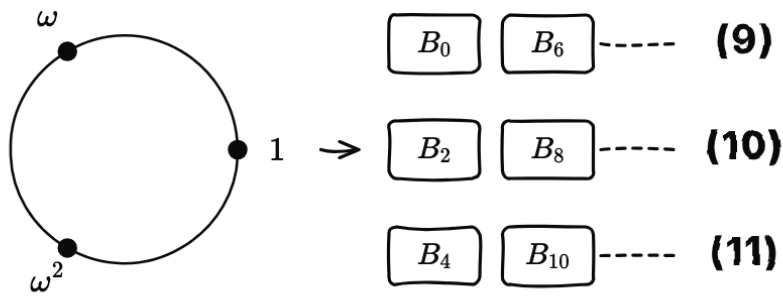
Half the terms of before

Try it yourself

Check equation eight with n equal to 4 on paper: the only terms are $c_2 B_{2/2}$ and the last one. With $c_2 = 6$ and $B_2 = 1/6$ you get $6 \times (1/6)/2 - 4/2^3 = 0.5 - 0.5 = 0$.

In section four Ramanujan pushes the same idea from two to three, using the cube roots of unity $1, \omega, \omega^2$, the three numbers whose cube is 1, with $1 + \omega + \omega^2 = 0$. From the identity $4 \sin x \sin x\omega \sin x\omega^2 = -(\sin 2x + \sin 2x\omega + \sin 2x\omega^2)$ and logarithmic differentiation, he gets three-way mixtures of cotangents, and now two of every three terms cancel. The survivors give equations nine, ten and eleven, for example

$3(B_0 + B_6x^6/6! + B_{12}x^{12}/12! + \dots) = -x(x^2/2! - x^8/8! + \dots) / (x^3/3! - x^9/9! + \dots)$, which keeps only every sixth B. For tiny x its right side is $-x(x^2/2)/(x^3/6) = -3$, matching $3B_0 = -3$. Equation ten keeps B two, B eight, B fourteen, and its leading term $x(x^4/24)/(x^3/6) = x^2/4$ matches $3B_2x^2/2 = x^2/4$; equation eleven keeps B four, B ten, B sixteen.



EQ. 4

EQ. 12

Three strands, six apart

Matching coefficients across these gives equation twelve: $c_3B_{n-3} - c_9B_{n-9} + c_{15}B_{n-15} - \dots = 0$, for odd n, closed by a last term $\frac{1}{6}n$ or $\frac{1}{3}n$ with a sign that depends on the remainder of n on division by 6. For n equal to 5 it is just $c_3B_2 = 5/3$, that is $10 \times 1/6 = 5/3$: steps of six, a third of the terms of equation four. Finally he divides equation ten by

x and differentiates, preparing a product formula for checking, which comes next.

Remember

Adding the cotangent and its hyperbolic twin cancels every second Bernoulli term, keeping B nought, B four, B eight; subtracting keeps B two, B six, B ten. De Moivre's theorem turns each half into a ratio of simple series, equations six and seven, and matching coefficients gives equation eight with only about a quarter of n terms. Cube roots of unity do the same in steps of three, so equations nine to eleven each keep every sixth B, and equation twelve needs only a third of the terms of the plain method. The leading terms of these ratios check the first values at once: B nought is minus one and B two is one sixth.

Questions

1. Why does adding the cotangent and the hyperbolic cotangent keep only B₀, B₄, B₈, ...?
 - a. Because the hyperbolic series has the same coefficients with every second sign flipped, so the B₂, B₆, B₁₀ terms cancel
 - b. Because the hyperbolic cotangent is zero
 - c. Because B₂, B₆, B₁₀ are themselves zero
 - d. Because De Moivre's theorem removes odd terms
2. What does the leading term of equation seven tell you?

- a. That $B_0 = -1$
 - b. That $B_2 = 1/6$, since $x \cdot (x^3/12) / (x^2/2) = x^2/6$ matches $B_2 x^2$
 - c. That $B_4 = 1/30$
 - d. That the series diverges
3. How many terms does equation eight need compared with equations two to four?
- a. About the same number
 - b. About a quarter of n instead of half of n , only every second earlier B
 - c. Only one term
 - d. Twice as many
4. Why do the cube roots of unity make two of every three terms cancel?
- a. Because $1 + \omega + \omega^2 = 0$, so a three-way mixture cancels all but every third term
 - b. Because ω is a real number
 - c. Because $\cot \omega = 0$
 - d. Because they are all equal to one

Answers

- 1. a)** The series of $\coth$ has the same coefficients with every second sign flipped, so adding the two cancels the terms whose index is not a multiple of four.
- 2. b)** For tiny x the right side is $x^2/6$ and the left side starts with $B_2 x^2$, so $B_2 = 1/6$.

3. b) Equations two to four had $\frac{1}{2}n$ terms, all the earlier B's; equation eight has about $\frac{1}{4}n$, only every second one.

4. a) The three cube roots add up to zero, so the three-way mixture of cotangents keeps only one strand of B's out of three.

Worked examples

Equation eight for $n = 6$ Check equation eight for $n = 6$ with $B_0 = -1$ and $B_4 = 1/30$. Here $n - 2 = 4$ is a multiple of 4, so the last term is $(6/2^4)(-1)^{((6-2)/4)}$.

The terms are $c_2B_4/2$ and $-c_6B_0/2^3$ with $c_2 = 15$ and $c_6 = 1$.

$$c_2B_4/2 = 15 \times (1/30)/2 = 0.25.$$

$$-c_6B_0/2^3 = -(-1)/8 = 0.125.$$

$$\text{The last term is } (6/16) \times (-1)^1 = -0.375.$$

The sum is $0.25 + 0.125 - 0.375 = 0$, as equation eight requires.

Answer: 0, so equation eight holds

Practice

1. Use the leading term of equation six to find B_0 .

Hint: For tiny x the right side is $-x \cdot x / (x^2/2)$.

Answer: $B_0 = -1$: the right side tends to -2 and the left side to $2B_0$.

2. Use equation twelve with $n = 5$ to find B_2 .

Hint: For $n = 5$ the equation is just $c_3B_2 = 5/3$, with $c_3 = 10$.

Answer: $B_2 = 1/6$: $10B_2 = 5/3$.

3. Use the leading term of equation ten to find B_2 again.

Hint: The right side starts with $x(x^4/24)/(x^3/6)$; the left side with $3B_2x^2/2$.

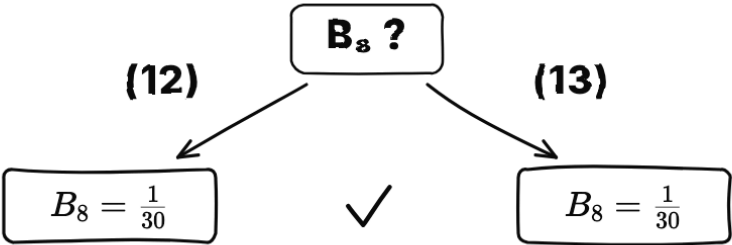
Answer: $B_2 = 1/6$: the right side is $x^2/4$, so $3B_2/2 = 1/4$.

A Formula to Check the Work, and the Table

An accountant who adds a long column twice, once down and once up, is not wasting time: two different routes that agree make a mistake very unlikely. Ramanujan builds the same safeguard into his method.

$$\frac{1}{3}(n+2)B_n = c_6B_{n-6}B_6 + c_{12}B_{n-12}B_{12} + \cdots$$

(13)



$$\frac{10}{3}B_8 = 28 \cdot \frac{1}{6} \cdot \frac{1}{42} = \frac{1}{9}$$

Two roads to B8

After differentiating equation ten, the derivative of the ratio turns out to be $1 - (x^2/2! - x^8/8! + \cdots)(x^4/4! - x^{10}/10! + \cdots) / (x^3/3! - x^9/9! + \cdots)^2$. Each of those two fractions is already known from equations

nine and ten, so the right side becomes $x^2 + 9(B_0 + B_6x^6/6! + \dots)$
 $(B_2x^2/2! + B_8x^8/8! + \dots)$. Matching the coefficient of x^n gives result
thirteen, valid when n minus 2 is a multiple of 6: $\frac{1}{3}(n+2)B_n =$
 $c_6B_{n-6}B_6 + c_{12}B_{n-12}B_{12} + c_{18}B_{n-18}B_{18} + \dots$. In words: a B in the
strand two, eight, fourteen equals a sum of products of earlier pairs.
Where it comes from: the derivative of a ratio of series can be written
through the ratios themselves, the way the derivative of the tangent is
one plus the tangent squared. For n equal to 8 there is one product:
 $(10/3)B_8 = 28 \times (1/6) \times (1/42) = 1/9$, so $B_8 = 1/30$. Ramanujan states
the division of labour plainly: equation twelve computes the B 's very
quickly, and equation thirteen may prove useful in checking. Both
have a third of the terms of equation four, so B_{24} needs only B_{18} , B_{12}
and B_6 .

Section five puts the method to work. Ramanujan wants to study the
 B 's by their properties, and for that he first needs a stock of values, so
he sets n equal to 3, 5, 7, 9 and so on in equation twelve. Each choice
gives a short equation, such as $B_8 - \frac{1}{3}B_2 = -1/45$. With $B_2 = 1/6$ this
is $B_8 = 1/18 - 1/45 = 3/90 = 1/30$, the same value result thirteen gave.
The next is $B_{10} - (5/2)B_4 = -1/132$, so $B_{10} = 1/12 - 1/132 = 5/66$.
And $B_{12} - 11B_6 = -4/455$ gives $B_{12} = 11/42 - 4/455 = 691/2730$.

$$B_8 - \frac{1}{3}B_2 = -\frac{1}{45}$$

$$B_8 = \frac{1}{18} - \frac{1}{45} = \frac{1}{30}$$

$$B_{10} = \frac{1}{12} - \frac{1}{132} = \frac{5}{66}$$

$$B_{12} = \frac{11}{42} - \frac{4}{455} = \frac{691}{2730}$$

$$\frac{11}{42} - \frac{4}{455}$$



0.2531136

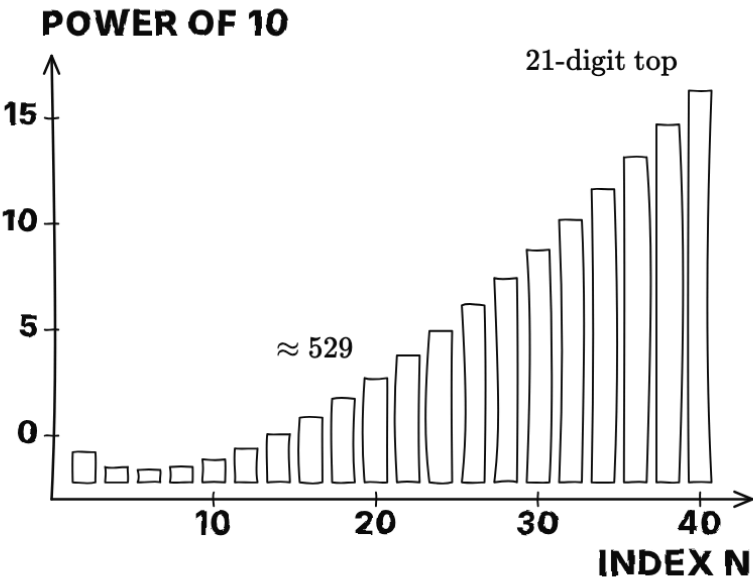
Three values by hand

Try it yourself

Take a calculator and check the last one: eleven divided by forty-two, minus four divided by four hundred and fifty-five. You should see $0.2619048 - 0.0087912 = 0.2531136$, and $691/2730$ gives the same digits.

Carrying on this way, he lists twenty values, from $B_2 = 1/6$ to B_{40} . They stay small at first, $B_{14} = 7/6$, then climb fast: $B_{20} = 174611/330$, about 529, and B_{40} has a numerator of 21 digits over a denominator of only 13530. The denominators stay modest while the numerators explode, and the table ends with the remark that the B's grow beyond

all bounds. These twenty fractions are the evidence from which the next sections draw their patterns.



Twenty values: small, then enormous

Remember

Differentiating equation ten and using equations nine and ten together gives result thirteen, which writes a B as a sum of products of earlier pairs. Equation twelve is for fast computation and equation thirteen for checking; both need only a third of the terms of the plain method, and for B eight both give one thirtieth. Putting n equal to 3, 5, 7 and so on into equation twelve, Ramanujan obtains a table of twenty B 's, small fractions at first, then numbers with numerators of twenty and more digits.

Questions

1. What is result thirteen used for, according to Ramanujan?
 - a. For computing the B 's fastest
 - b. For checking the calculations made with equation twelve
 - c. For finding the denominators
 - d. For proving that the B 's are positive
2. Why does Ramanujan compute a table of B 's in section five?
 - a. To study the B 's through their properties he first needs a stock of values
 - b. Because equation twelve cannot be used otherwise
 - c. To check the value of π
 - d. Because the table proves his later statements

3. What does the table show about the size of the B's?
 - a. They decrease towards zero
 - b. They stay below one
 - c. They stay small at first, then grow very fast while the denominators stay modest
 - d. They are all whole numbers

Answers

1. **b)** Equation twelve computes the B's very quickly, and equation thirteen may prove useful in checking.
2. **a)** He wants to study the B's by their properties, and for that he first needs a stock of values.
3. **c)** The values stay small at first and then climb fast; the denominators stay modest while the numerators explode.

Worked examples

B₈ two ways Find B₈ from result thirteen with $n = 8$, and again from the section-five equation $B_8 - \frac{1}{3}B_2 = -\frac{1}{45}$, using $B_2 = \frac{1}{6}$ and $B_6 = \frac{1}{42}$.

Result thirteen for $n = 8$ has one product: $(\frac{10}{3})B_8 = c_6B_2B_6$ with $c_6 = 28$.

$28 \times \frac{1}{6} \times \frac{1}{42} = \frac{1}{9}$, so $B_8 = (\frac{1}{9}) \times (\frac{3}{10}) = \frac{1}{30}$.

The section-five equation gives $B_8 = \frac{1}{3} \times \frac{1}{6} - \frac{1}{45} = \frac{1}{18} - \frac{1}{45}$.

$\frac{1}{18} - \frac{1}{45} = \frac{5}{90} - \frac{2}{90} = \frac{3}{90} = \frac{1}{30}$.

Answer: $B_8 = \frac{1}{30}$ both ways

Practice

1. From $B_{10} - (5/2)B_4 = -1/132$ and $B_4 = 1/30$, find B_{10} .

Hint: Five halves of one thirtieth is one twelfth.

Answer: $B_{10} = 1/12 - 1/132 = 11/132 - 1/132 = 10/132 = 5/66$.

2. From $B_{12} - 11B_6 = -4/455$ and $B_6 = 1/42$, find B_{12} as a fraction.

Hint: Use the common denominator $2730 = 42 \times 65 = 455 \times 6$.

Answer: $B_{12} = 11/42 - 4/455 = 715/2730 - 24/2730 = 691/2730$.

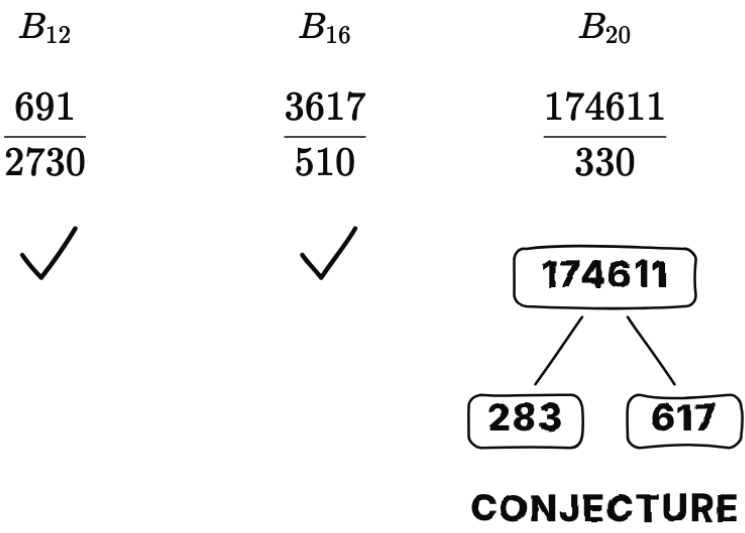
3. About how large is $B_{20} = 174611/330$?

Hint: Divide 174611 by 330.

Answer: About 529.1, since $330 \times 529 = 174570$ and the remainder is 41.

Patterns Read Off the Table

Look at the table again: $B_{12} = 691/2730$, $B_{16} = 3617/510$, $B_{20} = 174611/330$. Every denominator is even and divisible by three, and the denominators are small and tidy while the numerators look random. Ramanujan now states what he observes, for every even n other than zero, and he is careful to call these things observed: he promises reasons later, in section twelve.

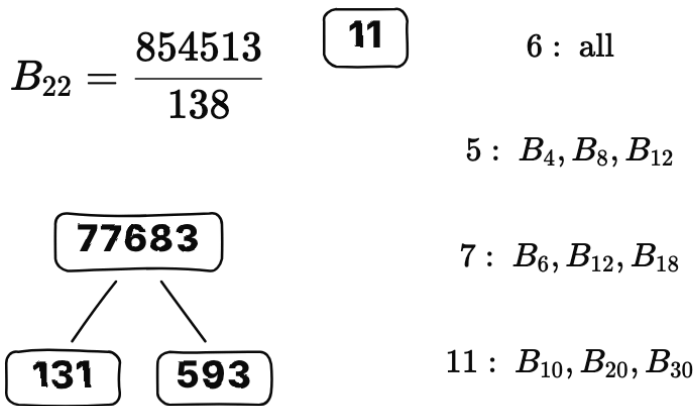


Three specimens, one broken rule

Statement fourteen: B_n is a fraction, and the numerator of B_n/n in lowest terms is a prime. For B_{12} that numerator is 691, a prime, and

for B_{16} it is 3617, also prime. But checking further breaks it: for B_{20} the numerator $174611 = 283 \times 617$, so it is not prime, and the editors of the collected papers list this statement as a conjecture.

Statement fifteen: the denominator of B_n contains 2 and 3 exactly once each, as 6, 30, 42 and 2730 all do. Statement sixteen: $2^n(2^n - 1)B_n/n$ is a whole number, and so $2(2^n - 1)B_n$ is an odd whole number. For n equal to 6: $2 \times 63 \times 1/42 = 3$, odd, as promised; for n equal to 12: $2 \times 4095 \times 691/2730 = 2073$, odd again. From this he deduces statement seventeen, that if n is not a multiple of 4 then $4(2^n - 1)B_n/n$ is odd: for n equal to 6 it is $4 \times 63/(42 \times 6) = 1$. Statement eighteen follows from fourteen: a prime dividing n but missing from the denominator divides the numerator, with a prime quotient.



Eleven goes to the top

His examples: 11 divides 22 but not the denominator 138 of B₂₂, so 11 divides its numerator; 13, 17, 19 do the same for B₂₆, B₃₄, B₃₈. The divisibility parts are right, but the claimed prime quotients do not all survive: for B₂₂ the quotient is $77683 = 131 \times 593$.

Section seven turns to denominators. All are divisible by 6; those of B₄, B₈, B₁₂ by 5; those of B₆, B₁₂, B₁₈ by 7; those of B₁₀, B₂₀, B₃₀ by 11. From this he infers statement nineteen: the denominator of B_n is the product of the primes that are one more than a factor of n, unity and n included. For B₂₄: the factors 1, 2, 3, 4, 6, 8, 12, 24 plus one give 2, 3, 4, 5, 7, 9, 13, 25; keep the primes 2, 3, 5, 7, 13, and the product is 2730. Odd factors other than unity can be skipped, because one more than an odd number is even, so not prime. His further examples include B₂₀, whose primes 2, 3, 5, 11 give 330, and B₇₂, whose denominator is 140100870.

A THEOREM

	1	2	3	4	6	8	12	24
+1 ↓	2	3	~~4~~	5	7	~~9~~	13	~~25~~

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$$

$$B_{20} : 2 \cdot 3 \cdot 5 \cdot 11 = 330$$

$$B_{30} : 2 \cdot 3 \cdot 7 \cdot 11 \cdot 31 = 14322$$

Statement nineteen at work

Try it yourself

With pencil and paper, find the denominator of B_{30} by the rule. Unity and the even factors are 1, 2, 6, 10, 30; one more gives 2, 3, 7, 11, 31, all prime, and their product is 14322, exactly the denominator in his table.

The denominator rule and statements fifteen and sixteen are in fact true; the editors point out that the rule behind them was known to Clausen and von Staudt, and the oddness statement to Euler.

Remember

From his table Ramanujan states patterns for the even-index B's: prime numerators after dividing by n , the factors 2 and 3 exactly once in every denominator, and whole and odd numbers built from $2^n - 1$. The denominator of B_n is the product of the primes one more than a factor of n ; for B twenty-four that gives 2730. These are stated as observations: the denominator and oddness patterns are true and were known before, but the prime-numerator claim fails, for example $174611 = 283 \times 617$.

Questions

1. How should the statements in section six be understood?
 - a. As proved theorems
 - b. As observations read off the table, with reasons promised later
 - c. As definitions
 - d. As results quoted from Edwards
2. Why does statement fourteen fail for $n = 20$?
 - a. Because B_{20} is an integer
 - b. Because the numerator 174611 equals 283×617 , so it is not prime
 - c. Because 20 is a multiple of 4
 - d. Because the denominator of B_{20} is not divisible by 3

3. What does statement nineteen say about the denominator of B_n ?
- a. It is $n + 1$
 - b. It is the product of the primes that are one more than a factor of n , unity and n included
 - c. It is always 6
 - d. It is the largest prime below n
4. Why can odd factors other than unity be skipped in the denominator rule?
- a. Because one more than an odd number is even, so not prime
 - b. Because odd factors never divide n
 - c. Because they give the numerator
 - d. Because they are always 1

Answers

- 1. b)** Ramanujan states what he observes and is careful to call these things observed, promising reasons later in section twelve.
- 2. b)** For B_{20} the numerator $174611 = 283 \times 617$, so it is not prime, and the editors list the statement as a conjecture.
- 3. b)** Statement nineteen: the denominator of B_n is the product of the primes that are one more than a factor of n , unity and n included.
- 4. a)** One more than an odd number is even, hence not prime, so it never enters the product.

Worked examples

Statement sixteen for $n = 12$ Check that $2(2^{12} - 1)B_{12}$ is an odd whole number, with $B_{12} = 691/2730$.

$$2^{12} - 1 = 4095.$$

$$2 \times 4095 = 8190, \text{ and } 8190/2730 = 3.$$

$$\text{So } 2(2^{12} - 1)B_{12} = 3 \times 691 = 2073.$$

2073 is odd, as statement sixteen says.

Answer: 2073, an odd whole number

Practice

- 1.** Use statement nineteen to find the denominator of B_{12} .

Hint: Unity and the even factors of 12 are 1, 2, 4, 6, 12; add one to each.

Answer: 2730: one more gives 2, 3, 5, 7, 13, all prime, and their product is 2730.

- 2.** Check statement seventeen for $n = 10$: is $4(2^{10} - 1)B_{10}/10$ an odd whole number? Use $B_{10} = 5/66$.

Hint: $2^{10} - 1 = 1023 = 3 \times 11 \times 31$.

Answer: Yes: $4 \times 1023 \times 5/(66 \times 10) = 20460/660 = 31$, which is odd.

- 3.** For $B_{22} = 854513/138$, check that 11 divides the numerator and find the quotient.

Hint: Divide 854513 by 11.

Answer: $854513 = 11 \times 77683$, so 11 divides it; the quotient 77683 is 131×593 , not a prime, contrary to the claim.

The Fraction Part, and a Number Pinned From Its Size

Suppose a friend tells you a mystery fraction is a bit more than six thousand, that its bottom is 138, and that its top is divisible by eleven. That is almost enough to name it exactly. This part shows how Ramanujan does exactly that with a Bernoulli number, and then brings in the tools that estimate its size.

$$(-1)^{n/2} \left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots \right) - (-1)^{n/2}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = \frac{47}{510}$$

$$3617 = 7 \cdot 510 + 47$$

$$1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$$

The fraction part from primes

First the fractional part, what is left of B_n after its whole part is removed. Statement twenty says: the fractional part of B_n is

$(-1)^{(n/2)}(1/p_1 + 1/p_2 + \dots) - (-1)^{(n/2)}$. In words: add up one over each prime in the denominator and subtract one, flipping the sign of the whole thing when half of n is odd. Where it comes from: any fraction whose denominator is a product of different primes can be split into simple pieces with one prime each, and the rule says these pieces are always plus or minus one over the prime. For $B_{16} = 3617/510$ the primes are 2, 3, 5, 17, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = 47/510$; indeed $3617 = 7 \times 510 + 47$. For B_{22} the sign flips: $1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = 17/138$.

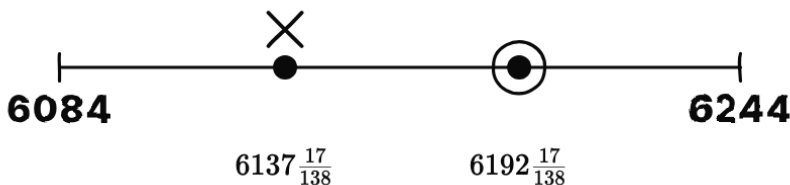
Statement twenty-one follows: if G is the greatest common measure and L the least common multiple of the denominators of B_m and B_n , then L/G is the denominator of $B_m - (-1)^{((m-n)/2)}B_n$. In particular, equal denominators make that difference a whole number: $B_{24} - B_{12}$ is an integer, since both have 2730 below, while $B_{10} + B_{20}$ has denominator $330/66 = 5$. He adds statement twenty-two: if n is a multiple of 4, the numerator of $B_n - 1/30$ is divisible by 20, as for $B_{12} - 1/30 = 600/2730 = 20/91$; otherwise the numerator of $B_n/n - 1/12$ is divisible by 5. And statement twenty-three: certain numbers like $2(2^{4n+2} - 1)B_{4n+2}/(2n + 1)$ are whole numbers of the form $30p + 1$; for n equal to 1 it is $2 \times 63 \times (1/42)/3 = 1$.

Section ten puts these together. Suppose all you know is that B_{22} lies between 6084 and 6244. Its fractional part must be $17/138$, its numerator must be divisible by 11, and $B_{22} - 11/6$ must be divisible by 5. Among the 160 whole numbers in that range only two survive: 6137 $17/138$ and 6192 $17/138$. Ramanujan picks the second because the first has a numerator divisible by 7 and 17 as well, and he expects the numerator divided by 11 to be prime. The answer, $854513/138$, is right, but the reason is shaky: $854513 = 11 \times 131 \times 593$, so the quotient is not prime either.

$$\frac{17}{138}$$

$$11 \mid \text{top}$$

$$5 \mid B - \frac{11}{6}$$



$$846923 = 7 \cdot 17 \cdot 7117$$

$$854513 = 11 \cdot 131 \cdot 593$$

Two survivors out of 160

Try it yourself

With a calculator, divide 846923, the numerator of the first candidate, by 7 and then by 17. Both divisions come out exact, which is why Ramanujan rejected that candidate.

To get such limits in the first place he needs the size of B_n , and section eleven supplies it: $B_n = (2 \cdot n! / (2\pi)^n)(1/1^n + 1/2^n + 1/3^n + \dots)$, equation twenty-four. In words: a Bernoulli number is a factorial divided by a power of 2π , times a sum that is barely more than one. For n equal to 2: $2 \times 2 / (4\pi^2) \times \pi^2 / 6 = 1/6$, using the famous sum of one over the squares. The same sum can be written as a product over

the primes, and it equals an integral: $B_n/(2n) = \int_0^\infty x^{n-1}/(e^{2\pi x} - 1) dx$, equation twenty-five, proved by expanding the fraction as a geometric series. He adds two similar integrals, one of them with a logarithm, equation twenty-six.

$$B_n = \frac{2n!}{(2\pi)^n} \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right)$$

$$\frac{2 \cdot 2}{4\pi^2} \cdot \frac{\pi^2}{6} = \frac{1}{6}$$

$$\frac{B_n}{2n} = \int_0^\infty \frac{x^{n-1}}{e^{2\pi x} - 1} dx$$

$$24.17714 - 10 + 0.91894 + 0.00833 = 15.10441$$

$$\log 3628800 = 15.10441$$

The size from a factorial

Then he takes logarithms in equation twenty-four and uses the classical expansion of $\log n!$: $(n + \frac{1}{2}) \log n - n + \frac{1}{2} \log 2\pi + B_2/(1 \cdot 2n) - B_4/(3 \cdot 4n^3) + \cdots$, equation twenty-seven, with the error after any term controlled by a factor θ between 0 and 1. For n equal to 10 the first terms give $24.17714 - 10 + 0.91894 + 0.00833 = 15.10441$, and $\log 10! = \log 3628800 = 15.10441$ agrees. So the whole part of B_n can be found from its size and the fraction part from statement twenty;

the snag, he notes, is that this needs logarithms to more decimal places than the tables give, a difficulty section thirteen removes.

Remember

The fractional part of B_n is plus or minus the sum of one over each prime of its denominator, minus one, so equal denominators make the signed difference of two B 's a whole number. Combined with divisibility rules, rough limits pin down a B exactly, as for $B_{22} = 854513/138$, though the prime-quotient argument used there is not sound. Equation twenty-four gives the size of B_n through a factorial, a power of two π and a sum of reciprocal powers, and the expansion of $\log n$ factorial turns that into a calculation that needs very precise logarithms.

Questions

1. What does statement twenty say about the fractional part of B_n ?
 - a. It is one over n
 - b. It is plus or minus the sum of one over each prime of the denominator, minus one
 - c. It is always one half
 - d. It equals the numerator divided by n
2. Why does Ramanujan reject the candidate $6137\ 17/138$ for B_{22} ?
 - a. Because it lies outside the limits

- b. Because its numerator is divisible by 7 and 17 as well as 11, and he expects the quotient by 11 to be prime
 - c. Because its fractional part is wrong
 - d. Because it is not divisible by 5
- 3. What does equation twenty-four tell us about the size of B_n ?
 - a. It is a factorial divided by a power of 2π , times a sum barely more than one
 - b. It is always less than one
 - c. It equals n squared
 - d. It is the number of primes below n
- 4. What practical difficulty does Ramanujan note in computing the whole part from equation twenty-seven?
 - a. It needs logarithms to more decimal places than the tables give
 - b. It needs the value of B_0
 - c. It only works for odd n
 - d. It requires a table of primes

Answers

- 1. b)** Add up one over each prime in the denominator and subtract one, flipping the sign of the whole thing when half of n is odd.
- 2. b)** He picks the second because the first has a numerator divisible by 7 and 17 as well, and he expects the numerator divided by 11 to be prime.
- 3. a)** A Bernoulli number is a factorial divided by a power of 2π , times a sum that is barely more than one.

4. a) The snag is that this needs logarithms to more decimal places than the tables give, a difficulty section thirteen removes.

Worked examples

B₂ from equation twenty-four Use equation twenty-four with $n = 2$ and the sum $1/1^2 + 1/2^2 + \cdots = \pi^2/6$ to find B_2 .

Equation twenty-four reads $B_2 = (2 \cdot 2!/(2\pi)^2)(1/1^2 + 1/2^2 + \cdots)$.

$2 \cdot 2! = 4$ and $(2\pi)^2 = 4\pi^2$, so the factor is $4/(4\pi^2) = 1/\pi^2$.

Multiplying by $\pi^2/6$ gives $1/6$.

Answer: $B_2 = 1/6$

Practice

1. Use statement twenty to find the fractional part of B_{28} , whose denominator $870 = 2 \times 3 \times 5 \times 29$, given that half of 28 is even.

Hint: Add one over each prime and subtract one.

Answer: $1/2 + 1/3 + 1/5 + 1/29 - 1 = (435 + 290 + 174 + 30 - 870)/870 = 59/870$.

2. Check statement twenty-two for $n = 10$: is the numerator of $B_{10}/10 - 1/12$ divisible by 5? Use $B_{10} = 5/66$.

Hint: $B_{10}/10 = 1/132$, and $1/12 = 11/132$.

Answer: Yes: $1/132 - 11/132 = -10/132 = -5/66$, whose numerator 5 is divisible by 5.

3. Using the first terms of equation twenty-seven, estimate $\log 10!$ and compare with the true value 15.10441.

Hint: Compute $10.5 \times \log 10 - 10 + 1/2 \log 2\pi + B_2/(2 \times 10)$ with $\log 10 = 2.302585$ and $1/2 \log 2\pi = 0.918939$.

Answer: $24.177144 - 10 + 0.918939 + 0.008333 = 15.104416$, agreeing with 15.10441.

A Proof That Hangs on One Claim

Add up $1/11^2 + 1/12^2 + 1/13^2 + \dots$ and you get about 0.0951663. There is a shortcut: $1/10 - 1/(2 \times 10^2) + (1/6)/10^3 = 0.0951667$, already right to six decimals. That shortcut is built from Bernoulli numbers, and in section twelve Ramanujan tries to use such series to prove the patterns he observed.

$$\frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \dots$$

$$\approx 0.0951663$$

$$\frac{1}{10} - \frac{1}{2 \cdot 10^2} + \frac{1/6}{10^3} = 0.0951667$$

$$\frac{1}{6} = B_2$$

$$\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots = \frac{1}{n} - \frac{1}{2n^2} + \frac{B_2}{n^3} - \frac{B_4}{n^5} + \frac{B_6}{n^7} - \dots$$

$$\text{remainder} = \int_0^\infty \dots \sinh^2 \pi n x \dots$$

$$\text{next term: } B_4/10^5 \approx 3 \cdot 10^{-7}$$

$$\dots + \frac{B_{2p} \theta}{n^{2p+1}}, \quad 0 < \theta < 1$$

A shortcut built from B's

His starting point is equation twenty-nine, the second derivative of $\log \Gamma(n+1)$, the logarithm of the factorial: $1/(n+1)^2 + 1/(n+2)^2 + \dots = 1/n - 1/(2n^2) + B_2/n^3 - B_4/n^5 + B_6/n^7 - \dots$. In words: the tail of

the sum of reciprocal squares, starting after n , is a series in powers of $1/n$ whose coefficients are the Bernoulli numbers. It comes from differentiating the expansion of $\log n!$ from the last part twice. With n equal to 10 it gives the shortcut above, and the next term, $B_4/10^5 \approx 0.0000003$, shows how fast the error shrinks.

The series does not converge forever, so he stops after some term and writes the rest as $B_{2p}\theta/n^{(2p+1)}$, with θ between 0 and 1. In words: the error after stopping is a fraction of the first term left out, never more. He backs this up with an integral formula for the remainder, involving $\sinh^2 \pi nx$, so that the size of the error is controlled.

Now the main idea. He writes down equation twenty-eight, a combination of the same tail, the term $1/(2x^2) - 1/x$, and a sum over primes: $-1/(6(x^3 - x)) + 1/(5(x^5 - x)) + 1/(7(x^7 - x)) + 1/(11(x^{11} - x)) + \dots$. He claims that this whole expression expands in powers of $1/x$ with integer coefficients, starting $1/x^{15} - 7/x^{17} + 55/x^{19} - 529/x^{21} + \dots$. If that were true, comparing coefficients would make $B_2 = 1/6$, $-B_4 = 1/6 + 1/5$, $B_6 = 1/6 + 1/7$ and so on all integers. Check them: $1/6 - 1/6 = 0$, $-1/30 - 5/30 + 6/30 = 0$, $1/42 - 7/42 + 6/42 = 0$, and for $B_{14} = 7/6$, $7/6 - 1/6 = 1$, exactly the first coefficient on the right. Writing $-1/6$ as $1/2 + 1/3 - 1$ turns this into statement twenty, the fractional-part rule, and statement twenty-one follows from it.

EQUATION (28)

$$\left[\frac{1}{(x+1)^2} + \dots \right] + \frac{1}{2x^2} - \frac{1}{x} - \frac{1}{6(x^3-x)} + \frac{1}{5(x^5-x)} + \frac{1}{7(x^7-x)} + \dots$$



$$= \frac{1}{x^{15}} - \frac{7}{x^{17}} + \frac{55}{x^{19}} - \frac{529}{x^{21}} + \dots \quad \textbf{INTEGERS?}$$

$$B_2 - \frac{1}{6}, \quad -B_4 - \frac{1}{6} + \frac{1}{5}, \quad B_6 - \frac{1}{6} + \frac{1}{7}, \dots \in \mathbb{Z}$$

$$\frac{1}{6} - \frac{1}{6} = 0$$

$$-\frac{1}{30} - \frac{5}{30} + \frac{6}{30} = 0$$

$$\frac{1}{42} - \frac{7}{42} + \frac{6}{42} = 0$$

$$B_{14} : \frac{7}{6} - \frac{1}{6} = 1$$

$$B_{10} : \frac{5}{66} - \frac{11}{66} + \frac{6}{66} = 0$$

$$-\frac{1}{6} = \frac{1}{2} + \frac{1}{3} - 1$$

STATEMENT 20

$$B_{10} - \frac{1}{6} + \frac{1}{11} ?$$

The claim and its checks

Try it yourself

Check one more coefficient yourself: $B_{10} - \frac{1}{6} + \frac{1}{11}$ with $B_{10} = \frac{5}{66}$. Over 66 it is $\frac{5}{66} - \frac{11}{66} + \frac{6}{66} = 0$, an integer, as the claim requires.

He repeats the trick twice. Replacing n by $\frac{1}{2}n$ in equation twenty-nine and subtracting half of the result gives the alternating tail $\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \dots$, equation thirty, whose coefficients are $(2^{(2p)} - 1)B_{2p}$; if twice that tail had integer coefficients, the second part of statement sixteen, the oddness of $2(2^n - 1)B_n$, would follow. From the first derivative of $\log n!$, equation thirty-one with Euler's constant γ ,

he gets equation thirty-two for $1/(n+2) - 1/(n+4) + 1/(n+6) - \dots$; integer coefficients there would give the first part of statement sixteen.

Every route ends with "if we can prove", and he never does. The editors note that these integer-coefficient claims are never proved, and that some steps use series that diverge, so the arguments are sketches; the results themselves are true and were known to von Staudt and Clausen and to Euler.

Remember

The tail of the sum of reciprocal squares has a series in powers of one over n with Bernoulli coefficients, and stopping it leaves an error controlled by a factor theta between 0 and 1. Ramanujan reduces the fractional-part rule and the oddness statements to claims that certain series have integer coefficients; the first coefficients do come out as 0, 0, 0 and then 1. Those claims are never proved in the paper, so section twelve is a route to proofs, not a proof.

Questions

1. What does equation twenty-nine express?
 - a. The tail of the sum of reciprocal squares as a series in powers of $1/n$ with Bernoulli coefficients
 - b. The value of π
 - c. The denominator of every B_n

- d. The sum of all Bernoulli numbers
2. What does the factor θ between 0 and 1 tell you?
 - a. That the error after stopping is a fraction of the first term left out
 - b. That the series converges to θ
 - c. That every coefficient is between 0 and 1
 - d. That the Bernoulli numbers are below 1
 3. Why is section twelve a route to proofs rather than a proof?
 - a. Because the claims about integer coefficients on which it rests are never proved
 - b. Because the coefficients checked by hand are wrong
 - c. Because the patterns are false
 - d. Because it uses no Bernoulli numbers

Answers

1. a) The tail of the sum of reciprocal squares, starting after n , is a series in powers of $1/n$ whose coefficients are the Bernoulli numbers.
2. a) The error after stopping is a fraction of the first term left out, never more.
3. a) Every route ends with 'if we can prove', and the integer-coefficient claims are never proved in the paper.

Worked examples

The tail for $n = 10$ Estimate $1/11^2 + 1/12^2 + 1/13^2 + \dots$ with the first terms of equation twenty-nine for $n = 10$, and say how large the next term is.

The first terms are $1/n - 1/(2n^2) + B_2/n^3$ with $B_2 = 1/6$.

For $n = 10$ that is $0.1 - 0.005 + 0.000167 = 0.095167$.

The next term is $-B_4/n^5 = -(1/30)/100000$, about -0.0000003 .

So the tail is about 0.0951663 , and the error after three terms is at most about three ten-millionths.

Answer: About 0.0951663

Practice

1. Check that the coefficient $B_6 - 1/6 + 1/7$ is an integer, with $B_6 = 1/42$.

Hint: Put everything over 42.

Answer: $1/42 - 7/42 + 6/42 = 0$, an integer.

2. Check that the coefficient $B_{14} - 1/6$ equals the first coefficient 1 on the right-hand side of equation twenty-eight, with $B_{14} = 7/6$.

Hint: Subtract one sixth from seven sixths.

Answer: $7/6 - 1/6 = 1$, which matches the term $1/x^{15}$.

3. In words, explain how writing $-1/6$ as $1/2 + 1/3 - 1$ turns the integer coefficients into the fractional-part rule.

Hint: Look at which reciprocals of primes appear.

Answer: Each integer coefficient is $\pm B_n$ plus one over each prime that belongs to it, with $1/2$ and $1/3$ coming from $-1/6$, minus one. Saying the total is a whole number means the fractional part of B_n is $\pm$ that sum of reciprocals minus one, which is statement twenty.

How Many Digits, and the Next Value From the Last

How many digits does 2^{100} have? You do not need to multiply it out: $100 \times \log_{10} 2 = 30.103$, so it has 31 digits, and since $10^{0.103} \approx 1.268$ it starts with 126. Ramanujan does the same for Bernoulli numbers far too large to write out.

$$2^{100}$$

$$100 \times \log_{10} 2 = 30.103 \Rightarrow 31 \text{ digits}$$

$$10^{0.103} \approx 1.268 \Rightarrow \text{starts } 126$$

$$\log_{10} B_n \approx \left(n + \frac{1}{2}\right) \log_{10} n - 1.2324743503 n + 0.700120$$

$$\text{true value} \approx \dots + 0.0362/n$$

$$\log_e B_n \approx \left(n + \frac{1}{2}\right) \log_e n - n(1 + \log_e 2\pi) + \frac{1}{2} \log_e 8\pi$$

$$(1 + \log_e 2\pi) \times 0.4342945 = 1.2324744$$

$$20.5 \times 1.30103 - 24.64949 + 0.70012 = 2.72175$$

$$+ 0.0018 = 2.72356, \quad 10^{2.72356} \approx 529.1$$

Digits from a logarithm

Result thirty-three: $\log_{10} B_n = (n + \frac{1}{2}) \log_{10} n - 1.2324743503 n + 0.700120$, the true value being larger by about $0.0362/n$. In words: the number of digits of B_n grows a little faster than n times the

number of digits of n . Where it comes from: take logarithms in equation twenty-four, replace $\log n!$ by its leading terms, giving $\log_e B_n \approx (n + 1/2) \log_e n - n(1 + \log_e 2\pi) + 1/2 \log_e 8\pi$, and multiply by $\log_{10} e = 0.4342944819$ to switch to base ten. Check the constant: $(1 + \log_e 2\pi) \times 0.4342945 = 2.8378771 \times 0.4342945 = 1.2324744$. For n equal to 20: $20.5 \times 1.30103 - 24.64949 + 0.70012 = 2.72175$; adding $0.0362/20 = 0.0018$ gives 2.72356, and $10^{2.72356} \approx 529.1$, exactly the size of B_{20} .

Section fourteen compares neighbours. Dividing equation twenty-four for n by the same for n minus 2 gives equation thirty-four: $B_n = (n(n-1)/(4\pi^2)) B_{n-2} (1 - (2^2 - 1)/(2^n - 1))(1 - (3^2 - 1)/(3^n - 1)) \dots$. In words: each B is the previous one times $n(n-1)/(4\pi^2)$, corrected by factors that are almost exactly 1 once n is moderately large. So, result thirty-five, the ratio B_n/B_{n-2} approaches $n(n-1)/(4\pi^2)$ very rapidly.

NEIGHBOURS

$$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2} \underbrace{\left(1 - \frac{3}{2^n - 1}\right) \left(1 - \frac{8}{3^n - 1}\right) \cdots}_{\approx 1}$$

$$\frac{B_n}{B_{n-2}} \rightarrow \frac{n(n-1)}{4\pi^2}$$

	B_{20}	$B_{20} \in (529, 530) \rightarrow \frac{174611}{330}$
	B_{18}	
	B_{16}	$6.079 \times \frac{7}{6} \approx 7.09 \Rightarrow 7 < B_{16} < 8$
	B_{14}	$B_{16} = 7 \frac{47}{510} = \frac{3617}{510}$

A ladder of ratios

This gives a ladder. Multiply the previous B by $n(n-1)/(4\pi^2)$ to learn between which two whole numbers the next one lies, then add the fractional part from statement twenty. For B_{16} : $(15 \times 16/(4\pi^2)) \times 7/6 = 6.079 \times 1.1667 \approx 7.09$, between 7 and 8; the fractional part is $47/510$, so $B_{16} = 7 \frac{47}{510} = 3617/510$. Ramanujan's table does this rung by rung up to B_{20} , which lies between 529 and 530 and comes out as $528 + 1/2 + 1/3 + 1/5 + 1/11 = 174611/330$.

Try it yourself

On a calculator, multiply $43867/798$, the value of B_{18} , by $380/(4\pi^2)$. You should get about 529.1, so B_{20} lies between 529 and 530, as the table says.

$$\frac{43867}{798} \times \frac{380}{4\pi^2}$$
$$\approx 529.1$$

$$529 < B_{20} < 530$$



digits from $\log_{10} B_n$

$$\frac{B_n}{B_{n-2}} \rightarrow \frac{n(n-1)}{4\pi^2}$$

whole part + fraction = exact

Try it on a calculator

Remember

Result thirty-three gives the logarithm of B_n , hence its number of digits and first digits, from n alone; for B twenty it predicts about 529. Equation thirty-four says each B is the previous one times $n(n-1)/(4\pi^2)$, times correction factors close to 1, so the ratio settles quickly. Multiplying up to find the two whole numbers a and b lies between, and adding the fractional part from statement twenty, gives its exact value rung by rung.

Questions

1. Where does result thirty-three come from?
 - a. From taking logarithms in equation twenty-four and keeping the leading terms of $\log n!$, then converting to base ten
 - b. From the table of denominators
 - c. From equation twelve
 - d. From the cube roots of unity
2. What does equation thirty-four say about B_n compared with B_{n-2} ?
 - a. B_n is B_{n-2} times $n(n-1)/(4\pi^2)$, times correction factors close to 1
 - b. B_n is twice B_{n-2}
 - c. B_n is B_{n-2} plus one
 - d. B_n is B_{n-2} divided by π

3. How does the ladder of section fourteen produce exact values?
 - a. By finding the two whole numbers a B lies between from the previous B, then adding the fractional part from statement twenty
 - b. By rounding to the nearest integer
 - c. By using only the denominator
 - d. By computing $\log_{10} e$

Answers

1. a) Take logarithms in equation twenty-four, replace $\log n!$ by its leading terms, and multiply by $\log_{10} e$ to switch to base ten.
2. a) Each B is the previous one times $n(n - 1)/(4\pi^2)$, corrected by factors that are almost exactly 1 once n is moderately large.
3. a) Multiply the previous B by $n(n - 1)/(4\pi^2)$ to learn between which two whole numbers the next lies, then add the fractional part from statement twenty.

Worked examples

The size of B₂₀ from result thirty-three Use result thirty-three with $n = 20$, including the correction $0.0362/n$, to estimate B₂₀.

$$(n + 1/2) \log_{10} n = 20.5 \times 1.30103 = 26.67112.$$

Subtract $1.2324743503 \times 20 = 24.64949$ and add 0.70012 : the result is 2.72175 .

Add the correction $0.0362/20 = 0.00181$ to get 2.72356 .

$10^{2.72356}$ is about 529.1 .

Answer: About 529.1 , matching $B_{20} = 174611/330$

Practice

1. Use the ratio $17 \times 18/(4\pi^2)$ and $B_{16} = 3617/510$, about 7.0922, to find between which whole numbers B_{18} lies.

Hint: $4\pi^2$ is about 39.478.

Answer: $306/39.478$ is about 7.751, and 7.751×7.0922 is about 54.97, so B_{18} lies between 54 and 55.

2. How many digits does 2^{100} have, using $\log_{10} 2 = 0.30103$?

Hint: Multiply by 100 and look at the whole part.

Answer: $100 \times 0.30103 = 30.103$, so 2^{100} has 31 digits.

A Giant Described, and Five-Way Sieves

B_{444} has more than six hundred digits, far too many to write out by hand in 1911. Yet Ramanujan describes it in detail, using only the tools of the earlier sections. Today a computer algebra system prints it in a second, so we can check every claim he makes.

$$B_{444} = \frac{N}{D}$$

$$(p-1) \mid 444 : \quad 2, 3, 5, 7, 13, 149, 223$$

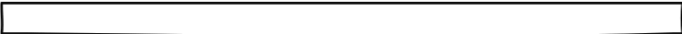
$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 149 \cdot 223 = 90709710$$

$$\text{fraction} = \frac{23975417}{90709710}$$

$$37 \mid N$$

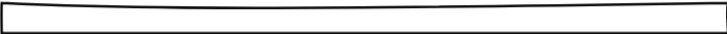
$$\log_{10} B_{444} \approx 630.2433$$

WHOLE PART: 631 DIGITS, 1751...



$$\log_{10} N \approx 630.2433 + 7.9577 = 638.2010$$

N: 639 DIGITS, 1588...



B444: size and fraction

Write $B_{444} = N/D$. The prime factors of D come from the denominator rule: the primes one more than a divisor of 444, which are 2, 3, 5, 7, 13, 149 and 223. Statement twenty then gives the fractional part $23975417/90709710$, and indeed $2 \times 3 \times 5 \times 7 \times 13 \times$

$149 \times 223 = 90709710$. Statement eighteen says 37, a prime dividing 444 but missing from D, divides N.

Result thirty-three gives $\log_{10} B_{444} \approx 630.2433$. In words: the whole part has 631 digits, and since $10^{0.2433} \approx 1.751$ it begins 1751. Adding $\log_{10} D = \log_{10} 90709710 \approx 7.9577$ gives $\log_{10} N \approx 638.2010$: the numerator has 639 digits and begins 1588.

Then the last digits. Statement twenty-two says the numerator of $B_{444} - 1/30$ is divisible by 20, because 444 is a multiple of 4. Write $[B_{444}]$ for the whole part of B_{444} ; then $[B_{444}] + 23975417/90709710 - 1/30 = [B_{444}] + 698392/3023657$. Its numerator $3023657 \times [B_{444}] + 698392$ must end in 0 with an even tens digit, which forces the whole part to end in 4 with an even digit before it; from that, N ends in 57 with an even third-from-last digit. A modern computation confirms it all: the whole part ends in 44, and N ends in 657. The one shaky ingredient is the claim that $N/37$ is prime, which rests on the unproved statement fourteen.

LAST DIGITS

$$20 \mid \text{top of } (B_{444} - \frac{1}{30})$$

$$[B_{444}] + \frac{23975417}{90709710} - \frac{1}{30} = [B_{444}] + \frac{698392}{3023657}$$

$$3023657k + 698392 \equiv 0 \pmod{20}$$

k ends ... $e4$

N ends ... $e57$

✓ **COMPUTER: ...44, ...657**

$$\frac{N}{37} \text{ prime? (rests on 14)}$$

$$7k + 2 \text{ ends in } 0$$

$$\Rightarrow 7k \text{ ends in } 8 \Rightarrow k \text{ ends in } 4$$

The last digits

Try it yourself

Check the arithmetic of the last step on paper: what must the last digit of k be for $3023657 \times k + 698392$ to be divisible by twenty? The last digit of the product is the last digit of $7k$, plus 2 must end in 0, so $7k$ ends in 8 and k ends in 4.

Section sixteen notes that the tangent and the cosecant can replace the cotangent. One result is equation thirty-six: $(4/3)B_n(2^n - 1) = c_6B_{n-6}(2^{n-6} - 1) - c_{12}B_{n-12}(2^{n-12} - 1) + \dots$, plus a last term $1/3n$ or $1/6n$ with a sign. For n equal to 6: $(4/3) \times (1/42) \times 63 = 2$, and the last term

is $\frac{1}{3} \times 6 = 2$. Equation thirty-seven does the same with factors $1 - 1/2^{n-4}$ and powers of 3; for n equal to 5 both sides equal 5/6.

COT

TAN

COSEC

$$\begin{aligned} \frac{4}{3}B_n(2^n-1) &= c_6B_{n-6}(2^{n-6}-1) - c_{12}B_{n-12}(2^{n-12}-1) + \dots \\ &+ \frac{1}{3}n \text{ or } \frac{1}{6}n \ (\pm) \end{aligned}$$

$$n = 6 : \quad \frac{4}{3} \cdot \frac{1}{42} \cdot 63 = 2 = \frac{1}{3} \cdot 6 \qquad \checkmark$$

$$(37) : \quad 1 - \frac{1}{2^{n-4}}, \quad 3^k$$

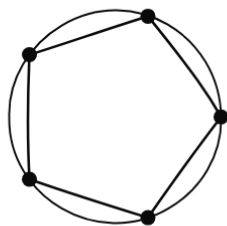
$$n = 5 : \quad \frac{5}{6} = \frac{5}{6} \qquad \checkmark$$

$$1, 2, 3\text{-interval} \rightarrow \omega^p = 1 \text{ (or } i) \rightarrow p\text{-interval}$$

Tangent, cosecant, intervals

Section seventeen names the family: equations from sections one, three and four are the one-, two- and three-interval formulas, and taking the p-th roots of unity, or of i when p is even, gives p-interval formulas. He works out p equal to 5 with the fifth roots 1, α, α², α³, α⁴. Adding their sines gives φ(x) = 5(x⁵/5! - x¹⁵/15! + ...): everything but every tenth power cancels, the five-way version of the seesaw. The golden ratio enters through α_n = ((1 + √5)/2)ⁿ + ((1 - √5)/2)ⁿ, which are whole numbers, 1, 3, 4, 7, 11 and so on. Equation thirty-eight, 5(Bo + B₁₀x¹⁰/10! + B₂₀x²⁰/20! + ...) = -x ((x⁴/4!)(1 + α₅) - ...) / ((x⁵/5!) (1 + α₅) - ...), keeps every tenth B; its leading term is

$-x(x^4/24)/(x^5/120) = -5$, matching $5B_0$. Equations thirty-nine to forty-two give the other four strands, and a similar identity with cosines and the function ψ gives still more.



$$\varphi(x) = 5\left(\frac{x^5}{5!} - \frac{x^{15}}{15!} + \cdots\right)$$

EVERY 10TH POWER

$$\alpha_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

1, 3, 4, 7, 11, ...



$$-x \cdot \frac{x^4/24}{x^5/120} = -5 = 5B_0 \quad \checkmark$$

FOUR MORE STRANDS, AND COSINES

The five-way sieve

Remember

For B₄₄₄ Ramanujan finds the fractional part, the number of digits and first digits of its whole part and numerator, and their last digits, all confirmed today; only the prime-quotient claim rests on an unproved statement. The tangent and cosecant give more relations of the same kind, checked for small n . The p -th roots of unity give p -interval formulas; for p equal to 5 the golden ratio's whole numbers appear and each strand keeps every tenth Bernoulli number.

Questions

1. How does Ramanujan find the denominator D of B_{444} ?
 - a. From the denominator rule: the primes one more than a divisor of 444, multiplied together
 - b. By computing B_{444} in full
 - c. From equation twelve
 - d. From the digit formula
2. Why does the whole part of B_{444} have 631 digits?
 - a. Because $\log_{10} B_{444}$ is about 630.2433
 - b. Because 444 has 3 digits
 - c. Because D has 8 digits
 - d. Because the numerator is divisible by 37
3. Which ingredient of the description of B_{444} is shaky?

- a. The claim that $N/37$ is prime, which rests on the unproved statement fourteen
- b. The number of digits
- c. The fractional part
- d. The last digits

Answers

1. a) The prime factors of D come from the denominator rule: the primes one more than a divisor of 444, which are 2, 3, 5, 7, 13, 149 and 223.
2. a) Result thirty-three gives $\log_{10} B_{444} \approx 630.2433$, so the whole part has 631 digits.
3. a) The one shaky ingredient is the claim that $N/37$ is prime, which rests on the unproved statement fourteen.

Worked examples

The digits of the numerator of B_{444} Given $\log_{10} B_{444} \approx 630.2433$ and $D = 90709710$, find the number of digits and the first digits of the numerator N .

$$\log_{10} N = \log_{10} B_{444} + \log_{10} D.$$

$$\log_{10} 90709710 \text{ is about } 7.9577.$$

$$\text{So } \log_{10} N \approx 630.2433 + 7.9577 = 638.2010.$$

The whole part 638 means 639 digits, and $10^{0.2010} \approx 1.588$ gives the first digits 1588.

Answer: 639 digits, beginning 1588

Practice

1. List the divisors of 444, add one to each, and keep the primes. What is their product?

Hint: $444 = 4 \times 3 \times 37$; its divisors are 1, 2, 3, 4, 6, 12, 37, 74, 111, 148, 222, 444.

Answer: One more gives 2, 3, 4, 5, 7, 13, 38, 75, 112, 149, 223, 445; the primes are 2, 3, 5, 7, 13, 149, 223, and their product is 90709710.

2. Check equation thirty-six for $n = 6$ with $B_6 = 1/42$.

Hint: The left side is $(4/3)B_6(2^6 - 1)$; the right side reduces to the last term, a third of six.

Answer: $(4/3) \times (1/42) \times 63 = 2$, and a third of 6 is 2, so both sides agree.

Four-Way Sieves, and the Editors' Verdict

The five-way sieve of the last part used the golden ratio; the four-way sieve uses its cousin, the square root of two. Ramanujan sets it up with two sequences, $a_n = (1 + 1/\sqrt{2})^n + (1 - 1/\sqrt{2})^n$ and $b_n = (1 + 1/\sqrt{2})^n - (1 - 1/\sqrt{2})^n$. In words: add or subtract the n -th powers of two numbers whose irrational parts cancel or double. For n equal to 2: $(1 + 1/\sqrt{2})^2 + (1 - 1/\sqrt{2})^2 = 2 \times (1 + 1/2) = 3$, because the cross terms $\pm\sqrt{2}$ cancel; so $a_2 = 3$. They obey rules such as $a_{m+n} = a_m a_n - a_{m-n}/2^n$, which let you build each from the previous ones.

$$1 \pm \frac{1}{\sqrt{2}}$$

$$a_n = \left(1 + \frac{1}{\sqrt{2}}\right)^n + \left(1 - \frac{1}{\sqrt{2}}\right)^n$$

$$b_n = \left(1 + \frac{1}{\sqrt{2}}\right)^n - \left(1 - \frac{1}{\sqrt{2}}\right)^n$$

CANCEL OR DOUBLE

$$(1 + \sqrt{2} + \frac{1}{2}) + (1 - \sqrt{2} + \frac{1}{2}) = 3 = a_2$$

$$a_{m+n} = a_m a_n - a_{m-n}/2^n$$

(43)–(46): strands spaced 8

$$4\left(B_0 + \frac{B_8 x^8}{8!} + \dots\right) = -x \frac{\frac{x^3}{3!} a_2 - \frac{x^{11}}{11!} a_6 + \dots}{\frac{x^4}{4!} a_2 - \frac{x^{12}}{12!} a_6 + \dots}$$

$$-x \cdot \frac{x^3/6}{x^4/24} = -4 = 4B_0 \quad \checkmark$$

Helpers from root two

With them he writes four formulas, equations forty-three to forty-six, one for each strand of Bernoulli numbers spaced eight apart. The first is $4(B_0 + B_8x^8/8! + B_{16}x^{16}/16! + \cdots) = -x ((x^3/3!)a_2 - (x^{11}/11!)a_6 + \cdots) / ((x^4/4!)a_2 - (x^{12}/12!)a_6 + \cdots)$. Check its leading term: $-x (x^3/6)/(x^4/24) = -4$, and the left side starts with $4B_0 = -4$, as it should. The others keep B_2, B_{10}, B_{18} , then B_4, B_{12}, B_{20} with a factor $4\sqrt{2}$, then B_6, B_{14}, B_{22} . That closes the paper: one idea, roots of unity cancelling terms, giving sieves of every size.

$$\begin{aligned}
 &B_0, \, B_8, \, B_{16}, \, \dots \\
 &B_2, \, B_{10}, \, B_{18}, \, \dots \\
 &B_4, \, B_{12}, \, B_{20}, \, \dots \quad (4\sqrt{2}) \\
 &B_6, \, B_{14}, \, B_{22}, \, \dots
 \end{aligned}$$

SIEVES OF EVERY SIZE

$$\begin{aligned}
 &\left(1 + \frac{1}{\sqrt{2}}\right)^3 + \left(1 - \frac{1}{\sqrt{2}}\right)^3 = 2\left(1 + \frac{3}{2}\right) = 5 \\
 &\checkmark \sqrt{2} \text{ terms cancel} \Rightarrow a_3 = 5
 \end{aligned}$$

Four strands, one more step

Try it yourself

Check the sequence yourself one step further: $a_3 = (1 + 1/\sqrt{2})^3 + (1 - 1/\sqrt{2})^3 = 2(1 + 3/2) = 5$. The terms with $\sqrt{2}$ cancel again, as they did for a_2 , so a_3 is a whole number too.

The book then adds a note from the editors of the collected papers, and it is worth reading closely, because it sorts everything we have met by its status. First, small clarifications: Ramanujan writes B_{2n} where most books write B_n , and c_k is the binomial coefficient $n!/(k!(n - k)!)$. Equation twelve holds for odd n greater than 1, with three forms of the last term according to the remainder of n on division by 6.

EDITORS

$$B_{2n} \leftrightarrow B_n$$

$$c_k = \frac{n!}{k! (n-k)!}$$

(12): odd $n > 1$

after (27), (29): invalid

×

20: VON STAUDT-CLAUSEN, 16: EULER

PROOF SKETCH COROLLARY CONJECTURE

16, 20, 21

17, 18

14, 15, 19

rest on (28), (30), (32)

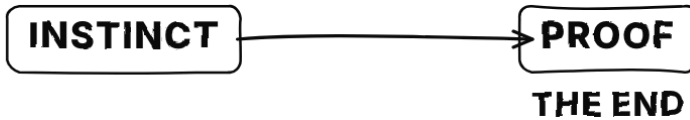
NEVER PROVED

The editors sort the claims

Then the substance. The derivations after equations twenty-seven and twenty-nine are invalid as they stand, because the series inside the integral diverges when x reaches n . The paper's principal results were already known: statement twenty is the theorem of von Staudt and Clausen, and statement sixteen, which follows from it, was known to Euler. Of the eight arithmetic statements, fourteen to twenty-one, proofs are indicated for sixteen, twenty and twenty-one, but they depend on equation twenty-eight and the matching claims for series thirty and thirty-two, which are never proved. Seventeen and eighteen are presented as corollaries, and fourteen, fifteen and nineteen as mere conjectures.

So the paper is, in the editors' words, interesting as a specimen of Ramanujan's earliest manner: fast methods, a famous theorem rediscovered, and bold patterns read from a table, with proofs still to come. This course has followed it from the cotangent to that verdict, and it ends here.

**FAST METHODS
REDISCOVERED
BOLD PATTERNS**



**STRANDS SPACED 8, FROM $1 \pm 1/\sqrt{2}$
KNOWN RESULTS, UNPROVED STEPS
COMPUTATION BEFORE RIGOUR**

Earliest manner

Remember

The four-interval formulas use sequences built from one plus or minus one over root two, and each isolates a strand of Bernoulli numbers spaced eight apart; their leading terms check against B_0 and B_2 . The editors' note clarifies notation and states the status of everything: some derivations are invalid as written, the main results were known to von Staudt, Clausen and Euler, the indicated proofs rest on unproved series claims, and three statements are conjectures, one of them false. Read with that note, the paper shows Ramanujan's earliest manner: powerful computation and instinct, before rigorous proof.

Questions

1. What do the four-interval formulas isolate?
 - a. Strands of Bernoulli numbers spaced eight apart, such as B_0 , B_8 , B_{16}
 - b. Every second Bernoulli number
 - c. Only B_2
 - d. The odd-index Bernoulli numbers
2. Why do the editors call the derivations after equations twenty-seven and twenty-nine invalid as they stand?
 - a. Because the series inside the integral diverges when x reaches n
 - b. Because they use the wrong value of π

- c. Because the integrals are zero
 - d. Because the B's are negative
3. Which statements do the editors say were stated merely as conjectures?
- a. Fourteen, fifteen and nineteen
 - b. Sixteen, twenty and twenty-one
 - c. Seventeen and eighteen
 - d. Twenty-two and twenty-three

Answers

1. **a)** Equations forty-three to forty-six give one formula for each strand of Bernoulli numbers spaced eight apart.
2. **a)** The derivations are invalid as they stand, because the series inside the integral diverges when x reaches n .
3. **a)** Seventeen and eighteen are presented as corollaries, and fourteen, fifteen and nineteen as mere conjectures.

Worked examples

The sequence a_n for small n Compute a_2 and a_3 , where $a_n = (1 + 1/\sqrt{2})^n + (1 - 1/\sqrt{2})^n$.

For $n = 2$, $(1 \pm 1/\sqrt{2})^2 = 1 \pm \sqrt{2} + 1/2$, and adding the two cancels $\pm\sqrt{2}$.

So $a_2 = 2 \times (1 + 1/2) = 3$.

For $n = 3$, $(1 \pm 1/\sqrt{2})^3 = 1 \pm 3/\sqrt{2} + 3/2 \pm 1/(2\sqrt{2})$, and adding cancels the terms with $\sqrt{2}$.

So $a_3 = 2 \times (1 + 3/2) = 5$.

Answer: $a_2 = 3$ and $a_3 = 5$

Practice

1. Check the leading term of equation forty-three: what does $-x (x^3/3!)a_2 / ((x^4/4!)a_2)$ equal for small x ?

Hint: The factors a_2 cancel.

Answer: $-x (x^3/6)/(x^4/24) = -4$, matching $4B_0 = -4$.

2. According to the editors, on what do the indicated proofs of statements sixteen, twenty and twenty-one depend?

Hint: Think of the series of section twelve.

Answer: On equation twenty-eight and the corresponding claims about the series thirty and thirty-two, which are never proved.