

Collected Papers of Srinivasa Ramanujan



Srinivasa Ramanujan · 1927

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A Family of Fractions in the Cotangent

Set a calculator to radians, type one tenth, take its tangent, and divide one tenth by the result. You get a number a hair below one: zero point nine nine six six six four four. Now try a different route: start from one, take away one hundredth divided by three, and take away one ten-thousandth divided by forty-five. The same digits come out. Those divisors, three and forty-five, are not accidents; they are built from a family of fractions called the Bernoulli numbers, and the first paper Ramanujan ever published, in 1911, is about them.

CALCULATOR

0.9966644

=

$$1 - \frac{0.01}{3} - \frac{0.0001}{45}$$

0.9966644

$$x \cot x = 1 - \frac{1}{6} \cdot \frac{(2x)^2}{2!} - \frac{1}{30} \cdot \frac{(2x)^4}{4!} - \dots$$

$$\text{zeroth} = -1$$

$$\text{first} = \frac{1}{6} \Rightarrow \frac{x^2}{3}$$

$$\text{second} = \frac{1}{30} \Rightarrow \frac{x^4}{45}$$

$$x = 0.1 : \quad \frac{x^2}{3} = \frac{0.01}{3}, \quad \frac{x^4}{45} = \frac{0.0001}{45}$$

Two roads, the same digits

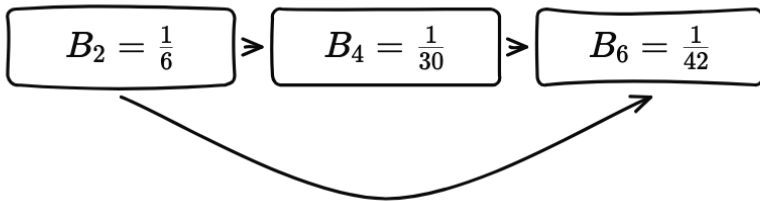
Ramanujan begins with a function from school trigonometry: x multiplied by the cotangent of x , where the cotangent is cosine divided by sine. Near zero this function can be written as one, minus a little multiple of x squared, minus a smaller multiple of x to the fourth, and so on through all even powers. He agrees to write each of those multiples using a special number: the first Bernoulli number, the second, the third, and so on, each divided by a factorial and multiplied by a power of two x . In his way of counting they are all positive, and to make the constant term one fit the same pattern he declares the zeroth Bernoulli number to be minus one. With the first one equal to one sixth, the first multiple becomes one sixth, halved, times four, which is one third; with the second equal to one thirtieth, the next becomes one forty-fifth. That is exactly the calculator's recipe: the cotangent series carries these fractions inside it.

How do you get the fractions out? Ramanujan writes the cotangent three different ways: as cosine over sine; as the sine of twice the angle over one minus the cosine of twice the angle; and as one plus that cosine over the same sine. Each top and bottom is a known power series. Multiply both sides of such an equation by the bottom, and you have two series that must be equal; two series are equal only when each power of x has the same coefficient on both sides. So matching, say, the coefficients of x to the sixth produces one equation linking the Bernoulli numbers, with counting numbers, the binomial coefficients, in front of them. He gets three such families of equations: one for odd powers with extra powers of two, one for even powers, and one for odd powers greater than one.

$$(x \cot x)^2 = -x^2 \left(1 + \frac{d \cot x}{dx} \right)$$

$$\frac{5}{2} B_4 = \frac{1}{2} \cdot 6 \cdot \left(\frac{1}{6} \right)^2 = \frac{1}{12}$$

$$\frac{d \tan x}{dx} = 1 + \tan^2 x$$



ALL EARLIER

Matching powers, one B at a time

Here is the even family at work. Matching the fourth power gives: six times the first Bernoulli number, plus one, minus two, equals zero; so the first one is one sixth. Matching the sixth power gives: fifteen times the second, minus fifteen sixths, minus one, plus three, equals zero; so the second is one thirtieth. Matching the eighth power gives one forty-second for the third. Every step works, but every step also needs all the earlier Bernoulli numbers, about half of the index's worth of terms, and Ramanujan says plainly that this grows tedious.

His first improvement squares the series. There is an exact identity: $x \cotangent x$, squared, equals minus x squared times one plus the derivative of the cotangent. Because both sides can be written with Bernoulli numbers, matching coefficients now gives a relation made

of products of two Bernoulli numbers, with the target multiplied by half of one more than its index. For the fourth power: five halves of the second Bernoulli number equals one half, times six, times one sixth squared, which is one twelfth; again the second one is one thirtieth. A similar relation comes from the fact that the derivative of the tangent is one plus the tangent squared.

His second improvement mixes the ordinary cotangent with its hyperbolic twin, which is the same function evaluated on an imaginary argument. Adding the two cancels every second term, so only the zeroth, the second, the fourth Bernoulli numbers and so on survive, every second one; subtracting them keeps the others. With De Moivre's theorem for powers of complex numbers, each of these half-families is again a ratio of two simple series, and matching coefficients gives equations that skip every other Bernoulli number. Instead of about half the earlier numbers, each new one now needs only about a quarter of the index's worth of terms, for example five instead of ten when the index is twenty. Similar results come from the tangent combined with the hyperbolic tangent.

COT B_0 B_2 B_4 B_6 B_8 B_{10}

COTH $+$ $-$ $+$ $-$ $+$ $-$

SUM $\square$ $\square$ $\square$ $\square$ $\square$ $\square$

BEFORE $\square$

NOW $\square$

The twin cancels every second term

Remember

Ramanujan's Bernoulli numbers are the positive fractions that sit in the series for x times the cotangent of x , with the zeroth one set to minus one. Writing the cotangent as a ratio of two series and matching each power of x gives equations that produce them one by one: one sixth, one thirtieth, one forty-second. Those plain equations each need all of the earlier numbers, which becomes slow. Squaring the series, or mixing the cotangent with its hyperbolic twin, gives relations that use only half as many earlier numbers.

Questions

1. Why does Ramanujan declare the zeroth Bernoulli number to be minus one?
 - a. Because all his Bernoulli numbers are negative
 - b. So that the constant term one of the series fits the same pattern as the other terms
 - c. Because the cotangent of zero is minus one
 - d. So that the equations have fewer terms
2. Why does matching the coefficients of two series give an equation?
 - a. Because two series can be equal only when each power of x has the same coefficient on both sides
 - b. Because coefficients are always whole numbers
 - c. Because every series equals the cotangent
 - d. Because the sine and cosine series are identical
3. What is the drawback of the plain equations from the three ways of writing the cotangent?
 - a. They give wrong values after the third number
 - b. Every step needs all the earlier Bernoulli numbers, so the work grows
 - c. They work only for odd powers
 - d. They need a calculator
4. What does mixing the cotangent with its hyperbolic twin achieve?

- a. It doubles every term
- b. It cancels every second term, so each half of the family can be found from fewer earlier numbers
- c. It removes the need for Bernoulli numbers
- d. It makes the series finite

Answers

1. **b)** In his counting all of them are positive, and to make the constant term one fit the same pattern he declares the zeroth one to be minus one.
2. **a)** After multiplying by the bottom you have two series that must be equal, and they are equal only when each power of x has the same coefficient.
3. **b)** Every step works, but every step also needs all the earlier Bernoulli numbers, and Ramanujan says this grows tedious.
4. **b)** Adding the two cancels every second term, and the resulting equations need only about a quarter of the index's worth of terms instead of half.

Worked examples

The first two Bernoulli numbers by matching powers Using the even family of equations and the zeroth number minus one, find the first and second Bernoulli numbers.

Matching the fourth power gives six times the first number, minus the zeroth number, minus two, equal to zero.

With the zeroth number minus one this is six times the first, plus one, minus two, equal to zero, so the first is one sixth.

Matching the sixth power gives fifteen times the second, minus fifteen times the first, plus the zeroth, plus three, equal to zero.

That is fifteen times the second, minus fifteen sixths, minus one, plus three, equal to zero, so fifteen times the second is one half.

Answer: The first is one sixth and the second is one thirtieth.

Practice

1. Check the calculator recipe: with the first Bernoulli number one sixth, what multiple of x squared appears in the series for x times the cotangent of x ?

Hint: The multiple is the number divided by two factorial, times two squared.

Answer: One third: one sixth divided by two, times four, is four twelfths, which is one third, and it is subtracted.

2. Use the squared-series relation for the fourth power to recover the second Bernoulli number from the first, one sixth.

Hint: Five halves of the second number equals one half, times six, times the first number squared.

Answer: One thirtieth: one half times six times one thirty-sixth is one twelfth, and one twelfth divided by five halves is one thirtieth.

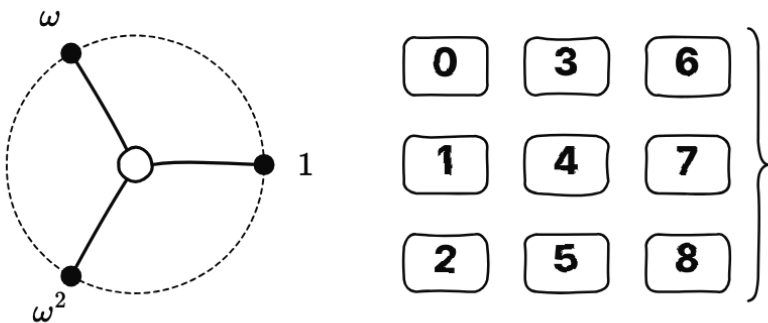
3. When the index is twenty, how many earlier terms does the plain even equation use, and how many does the hyperbolic shortcut use?

Hint: The plain equations use about half the index's worth of terms, the shortcut about a quarter.

Answer: About ten for the plain equation and about five for the shortcut, so the shortcut needs only half as many earlier numbers.

Steps of Six

Picture three friends pulling on a ring with ropes set at equal angles, a third of a full turn apart. If they pull equally hard, the ring does not move: the three pulls cancel exactly. Ramanujan's next shortcut uses three numbers arranged like those three ropes, and the cancelling is what saves him work.



$$3 \times (-1) = -3$$

$$\text{first} = \frac{1}{6}$$

Three ropes, three strands

The numbers are the three cube roots of one: one itself, and two complex numbers, usually called omega and omega squared, each a third of a turn around the circle from the others. Each of them,

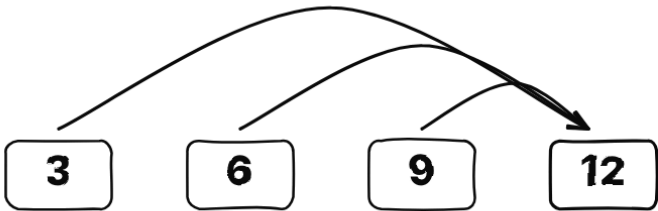
multiplied by itself three times, gives one, and the three of them add up to zero, like the three ropes. In the last part, mixing the cotangent with its hyperbolic twin made every second Bernoulli number cancel; feeding the cotangent all three cube roots makes two terms out of every three cancel.

He begins with an identity about sines: four times the sine of x , times the sine of $x\omega$, times the sine of $x\omega^2$, equals minus the sum of the sines of twice those three angles. Taking logarithms and differentiating turns that product into a sum of three cotangents, each weighted by its cube root. Halving the angle and following the same steps as before, he gets three new equations, each saying that a short series of Bernoulli numbers equals a ratio of two simple series. The first keeps only the zeroth, the third, the sixth Bernoulli numbers, every third one. The second keeps the first, the fourth, the seventh; the third keeps the second, the fifth, the eighth. So the whole family is split into three interleaved strands, three places apart, and each strand can be handled on its own.

You can check the first terms directly. In the first equation, for a tiny x the right side is minus x , times x squared over two, divided by x cubed over six, which is minus three; the left side starts with three times the zeroth Bernoulli number, and three times minus one is minus three. In the second, the right side starts as x squared over four, and the left side as three times the first Bernoulli number, halved, times x squared; that gives one sixth again.

Matching coefficients across the three equations gives his twelfth result: a short alternating sum of Bernoulli numbers, spaced six apart, with counting numbers in front, closed by one last term that is a sixth or a third of the index. For the index five it is as short as it can be: ten

times the first Bernoulli number equals five thirds, so the first is one sixth. In general each new number now needs only a third of the terms the plain method needed: the twelfth comes from just the ninth, the sixth and the third.



A THIRD OF THE WORK

$$10 \times \frac{1}{6} = \frac{5}{3}$$

$$\frac{1}{3}(n + 2) B_n = \sum \text{pairs}$$

$$\frac{10}{3} B_8 = 28 \cdot \frac{1}{6} \cdot \frac{1}{42} = \frac{1}{9}$$

Six places at a time

Then comes a second gift: a way to check the work. Dividing the second equation by x and differentiating, and using the first and second equations together, he finds a relation in which the series of the second strand, with extra multipliers, equals x squared plus nine times the product of the first two series. Matching coefficients gives his thirteenth result: one third of the index plus two, times a Bernoulli number, equals a sum of products of pairs, again spaced six apart. Try it for the index eight: one third of ten, times the fourth Bernoulli

number, should equal twenty-eight times one sixth times one forty-second, which is one ninth; so the fourth is one ninth times three tenths, one thirtieth, as before. Ramanujan says plainly what each is for: the twelfth result computes the numbers very quickly, and the thirteenth is useful for checking the calculations, the way an accountant adds a column twice by different routes.

Remember

The three cube roots of one add up to zero, and feeding them into the cotangent makes two of every three Bernoulli terms cancel. That splits the Bernoulli numbers into three strands, six places apart, each given by a ratio of simple series whose first terms check the known values at once. Matching coefficients gives a fast recurrence that needs only a third of the earlier terms of the plain method, and a product relation that serves as an independent check.

Questions

1. Why do the three cube roots of one make two of every three terms cancel?
 - a. Because they add up to zero, like three equal pulls on a ring
 - b. Because each of them equals one
 - c. Because they are all negative
 - d. Because they make the cotangent zero

2. What is the twelfth result good for, and what is the thirteenth good for?
 - a. The twelfth checks the work, the thirteenth computes the numbers
 - b. The twelfth computes the numbers very quickly, the thirteenth is useful for checking
 - c. Both prove that the numbers are fractions
 - d. Both give the denominators
3. Compared with the plain method, how many earlier terms does the new recurrence need?
 - a. The same number
 - b. Only a third of them
 - c. Twice as many
 - d. None at all

Answers

1. **a)** Each multiplied by itself three times gives one, and the three of them add up to zero, so the three-way mixture cancels two terms in every three.
2. **b)** Ramanujan says the twelfth result computes the numbers very quickly, and the thirteenth is useful for checking the calculations.
3. **b)** Each new number now needs only a third of the terms the plain method needed.

Worked examples

Checking with the product relation Use the thirteenth result for the index eight, with the first Bernoulli number one sixth and the third one forty-second, to find the fourth.

For the index eight only one product appears: twenty-eight, times the first number, times the third number.

Twenty-eight times one sixth times one forty-second is twenty-eight over two hundred and fifty-two, which is one ninth.

The left side is one third of ten, that is ten thirds, times the fourth number.

So the fourth number is one ninth divided by ten thirds, which is three ninetieths, one thirtieth.

Answer: The fourth Bernoulli number is one thirtieth.

Practice

1. Using the first cube-root equation, why must the zeroth Bernoulli number be minus one?

Hint: For a tiny x the right side is minus x , times x squared over two, divided by x cubed over six.

Answer: That ratio is minus three, and the left side starts with three times the zeroth number, so the zeroth number is minus one.

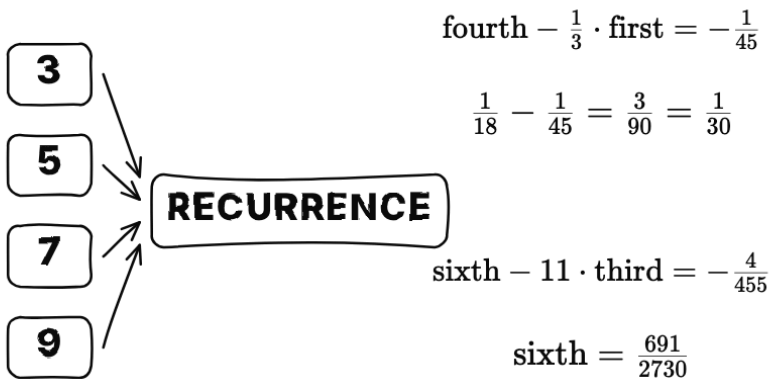
2. Using the twelfth result for the index five, find the first Bernoulli number.

Hint: For the index five it reads: ten times the first number equals five thirds.

Answer: One sixth, because five thirds divided by ten is one sixth.

Twenty Fractions and the Patterns in Them

A naturalist who wants to find the laws of bird migration first needs many sightings written down. Ramanujan works the same way: before stating any law about the Bernoulli numbers, he collects specimens. He says so directly: to know their properties, it is convenient to calculate a few of them first.

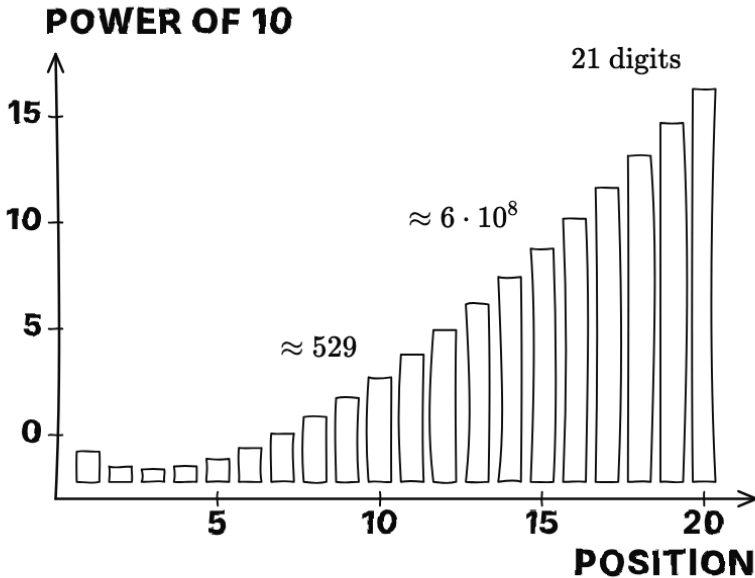


Specimens from the recurrence

His tool is the fast recurrence from the last part. He feeds it the odd numbers three, five, seven, nine, and so on, one after another, and

each gives a short equation. For example, one of them says that the fourth Bernoulli number, minus one third of the first, equals minus one forty-fifth. Since the first is one sixth, a third of it is one eighteenth, and one eighteenth minus one forty-fifth is three ninetieths, which is one thirtieth. Another says that the sixth, minus eleven times the third, equals minus four over four hundred and fifty-five; with the third equal to one forty-second, the sixth comes out as six hundred and ninety-one over two thousand seven hundred and thirty.

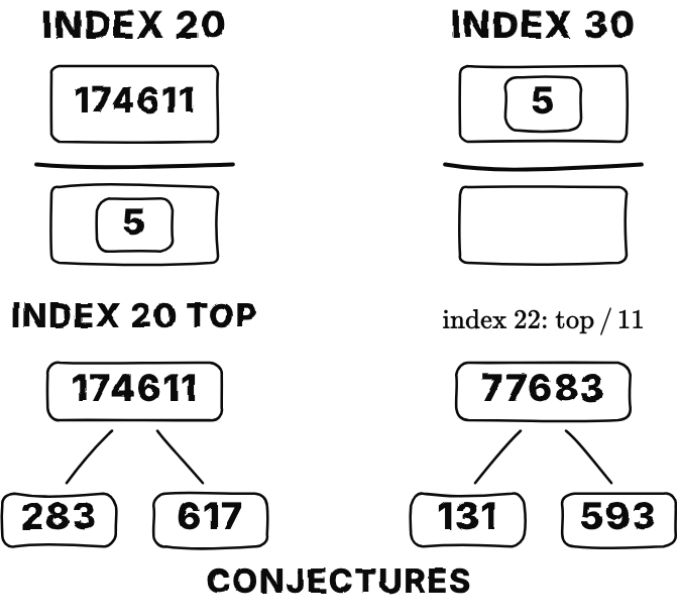
Going on like this, he lists twenty Bernoulli numbers. They stay small for a while, one sixth, one thirtieth, one forty-second, one thirtieth again, five sixty-sixths, then they climb fast. The tenth is about five hundred and twenty-nine, the fifteenth about six hundred million, and the twentieth has a numerator of twenty-one digits over a denominator of only thirteen thousand five hundred and thirty. The denominators stay modest while the numerators explode, and he notes that the sequence grows without bound.



Twenty specimens

Then he looks at the specimens and states what he observes, for every Bernoulli number with an even index other than zero. First: after dividing the number by its index and reducing, the numerator is a prime. Second: the denominator always contains two and three, each exactly once. Third: multiplying the number by two raised to the index, times one less than that power of two, and dividing by the index, gives a whole number; so twice one less than that power of two, times the number, is an odd whole number. From these he draws two consequences: one about which power of two remains in a denominator, and one saying that a prime dividing the index but missing from the denominator must divide the numerator, with a prime quotient.

His examples make the last idea vivid. Eleven divides the index twenty-two but is not in the denominator of that number, one hundred and thirty-eight, so eleven divides the numerator; the same happens with thirteen for the index twenty-six, seventeen for thirty-four and nineteen for thirty-eight. Five divides both twenty and thirty; it sits in the denominator for twenty but not for thirty, so it appears in the numerator for thirty instead. In his words, a prime appearing in the index that is not found in the denominator will appear in the numerator, and the other way round.



Where five goes, and two slips

These are observations, and not all of them survive checking. The numerator for the index twenty, one hundred and seventy-four thousand six hundred and eleven, which he calls prime, is two

hundred and eighty-three times six hundred and seventeen. The quotient for the index twenty-two, seventy-seven thousand six hundred and eighty-three, is one hundred and thirty-one times five hundred and ninety-three. The editors of the collected papers record that the first two of these statements, and the denominator rule of the next section, were stated only as conjectures; the claim about two and three in every denominator is nevertheless true.

Remember

To find properties of the Bernoulli numbers, Ramanujan first computes twenty of them with his fast recurrence, from one sixth up to a number with a twenty-one-digit numerator. From that table he reads off patterns: a prime numerator after dividing by the index, two and three exactly once in every denominator, and certain multiples that are always odd whole numbers. He links primes in the index to the numerator and the denominator: a prime of the index shows up in one of the two. Some of these patterns are true theorems, but the claim of a prime numerator fails, for example for the index twenty, whose numerator is two hundred and eighty-three times six hundred and seventeen.

Questions

1. Why does Ramanujan compute twenty Bernoulli numbers before stating any properties?

- a. Because, like a naturalist, he needs specimens before looking for laws
 - b. Because the journal required a table
 - c. Because the properties can only be proved for twenty of them
 - d. Because he wanted to show that the recurrence fails
2. What does Ramanujan observe about the denominators of the Bernoulli numbers with even index?
- a. They are always prime
 - b. They always contain two and three, each exactly once
 - c. They grow as fast as the numerators
 - d. They are always powers of two
3. Which observation fails when checked, as the example of the index twenty shows?
- a. That two and three divide every denominator
 - b. That the numerator, after dividing by the index, is a prime
 - c. That the numbers grow without bound
 - d. That five appears in the denominator for the index twenty

Answers

- 1. a)** He says that to know their properties it is convenient to calculate a few of them first, collecting specimens before stating laws.
- 2. b)** His second observation is that the denominator always contains two and three, each exactly once.
- 3. b)** The numerator for the index twenty, one hundred and seventy-four thousand six hundred and eleven, is two hundred and eighty-three times six hundred and seventeen, so it is not prime.

Worked examples

A value from the recurrence One equation of the recurrence says that the fourth Bernoulli number, minus one third of the first, equals minus one forty-fifth. With the first equal to one sixth, find the fourth.

One third of one sixth is one eighteenth.

So the fourth equals one eighteenth minus one forty-fifth.

In ninetieths, that is five ninetieths minus two ninetieths, three ninetieths.

Three ninetieths is one thirtieth.

Answer: One thirtieth

Practice

1. Check the claim of a prime numerator for the index twenty: does two hundred and eighty-three times six hundred and seventeen give the numerator?

Hint: Multiply two hundred and eighty-three by six hundred, then by seventeen, and add.

Answer: Yes: two hundred and eighty-three times six hundred is one hundred and sixty-nine thousand eight hundred, and two hundred and eighty-three times seventeen is four thousand eight hundred and eleven; together one hundred and seventy-four thousand six hundred and eleven, so the numerator is not prime.

2. In your own words, what does Ramanujan's rule say about a prime that divides the index?

Hint: Think of the examples with eleven for the index twenty-two and five for the index thirty.

Answer: Such a prime shows up either in the denominator or in the numerator of that Bernoulli number. If it is missing from the

denominator, as eleven is for the index twenty-two, it must divide the numerator, and the other way round.

A Recipe for Denominators

Here is a game you can play with a pencil. Take the number twenty-four and write down everything that divides it: one, two, three, four, six, eight, twelve, twenty-four. Add one to each: two, three, four, five, seven, nine, thirteen, twenty-five. Cross out every number that is not prime, and multiply what is left: two times three times five times seven times thirteen is two thousand seven hundred and thirty. Now look at Ramanujan's table: the Bernoulli number with index twenty-four has exactly that denominator.

DIVISORS OF 24

	1	2	3	4	6	8	12	24
+1 ↓	2	3	~~4~~	5	7	~~9~~	13	~~25~~

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$$

$$B_{24} = \frac{236364091}{2730}$$

$$20 : 2 \cdot 3 \cdot 5 \cdot 11 = 330 \qquad 30 : 2 \cdot 3 \cdot 7 \cdot 11 \cdot 31 = 14322$$

The pencil game for 24

That is his denominator rule. He noticed that all the denominators are divisible by six, those for indices that are multiples of four by five, those for multiples of six by seven, those for multiples of ten by eleven, and he inferred the general recipe: the denominator is the product of the primes that are one more than a divisor of the index, counting one and the index itself. Odd divisors other than one can be skipped, since one more than an odd number is even and so not prime. For the index twenty the recipe gives two, three, five and eleven, which multiply to three hundred and thirty, the denominator in the table; for the index thirty it gives two, three, seven, eleven and thirty-one, which multiply to fourteen thousand three hundred and twenty-two.

Next he looks at the fractional parts, what is left after removing the whole part. He splits each one into simple pieces and finds a rule: the fractional part is the sum of one over each prime in the denominator, minus one, with the whole expression taking a plus or minus sign that alternates with the index. For the index sixteen the primes are two, three, five and seventeen; a half plus a third plus a fifth plus a seventeenth, minus one, is forty-seven over five hundred and ten. And the Bernoulli number with index sixteen, three thousand six hundred and seventeen over five hundred and ten, is exactly seven plus forty-seven over five hundred and ten. For the index twenty-two the signs flip: one, minus a half, minus a third, minus a twenty-third, is seventeen over one hundred and thirty-eight.

FRACTIONAL PART

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = \frac{47}{510}$$

$$\frac{3617}{510} = 7 + \frac{47}{510}$$

$$1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$$

The leftover part

From this he draws a consequence about pairs: if two Bernoulli numbers have the same denominator, then their difference, or their sum when the signs require it, is a whole number. The indices twelve and twenty-four share the denominator two thousand seven hundred and thirty, so the difference of those two numbers is a whole number; the same holds for sixteen and thirty-two. For the indices ten and twenty the denominators differ only by a five, and their sum has a denominator of exactly five.

He adds two more observations about remainders. When the index is a multiple of four, the number minus one thirtieth has a numerator divisible by twenty: for the index twelve, six hundred and ninety-one over two thousand seven hundred and thirty minus one thirtieth is

twenty over ninety-one, and twenty is indeed the numerator. When the index is not a multiple of four, the number divided by the index, minus one twelfth, has a numerator divisible by five: for the index ten this gives minus five over sixty-six. Finally, certain whole numbers built from these Bernoulli numbers and powers of two always leave a remainder of one when divided by thirty; for the index two the first of them is just one, which is thirty times nothing plus one.

Remember

Ramanujan's denominator rule: take every divisor of the index, add one, keep the primes, and multiply; for twenty-four that gives two thousand seven hundred and thirty. The fractional part of each Bernoulli number is the sum of one over each prime of its denominator, minus one, with an alternating sign; for the index sixteen it is forty-seven over five hundred and ten. So two Bernoulli numbers with the same denominator differ by a whole number, and further rules restrict their numerators by divisibility by twenty or five and remainders of one on division by thirty.

Questions

1. How does Ramanujan's denominator rule find the denominator for a given index?
 - a. Take the divisors of the index, add one to each, keep the primes, and multiply them

- b. Multiply all primes below the index
 - c. Add one to the index and square it
 - d. Take the largest prime divisor of the index
2. What does the fractional-part rule give for the index sixteen?
- a. One seventeenth
 - b. Forty-seven over five hundred and ten
 - c. Seven
 - d. Seventeen over one hundred and thirty-eight
3. Why is the difference of the Bernoulli numbers with indices twelve and twenty-four a whole number?
- a. Because both are whole numbers
 - b. Because they share the same denominator, so their fractional parts cancel
 - c. Because twenty-four is twice twelve
 - d. Because both numerators are prime

Answers

- 1. a)** The denominator is the product of the primes that are one more than a divisor of the index, counting one and the index itself.
- 2. b)** A half plus a third plus a fifth plus a seventeenth, minus one, is forty-seven over five hundred and ten.
- 3. b)** If two Bernoulli numbers have the same denominator, their difference, or sum when the signs require it, is a whole number; twelve and twenty-four share two thousand seven hundred and thirty.

Worked examples

The denominator for the index twenty Use the denominator rule to find the denominator of the Bernoulli number with index twenty.

The divisors of twenty are one, two, four, five, ten and twenty.

Odd divisors other than one can be skipped, so use one, two, four, ten and twenty.

Adding one gives two, three, five, eleven and twenty-one.

Twenty-one is not prime, so keep two, three, five and eleven, and multiply: three hundred and thirty.

Answer: Three hundred and thirty

Practice

1. Check the rule about numerators divisible by twenty for the index twelve: what is six hundred and ninety-one over two thousand seven hundred and thirty, minus one thirtieth?

Hint: One thirtieth is ninety-one over two thousand seven hundred and thirty.

Answer: Six hundred over two thousand seven hundred and thirty, which reduces to twenty over ninety-one, and twenty is divisible by twenty.

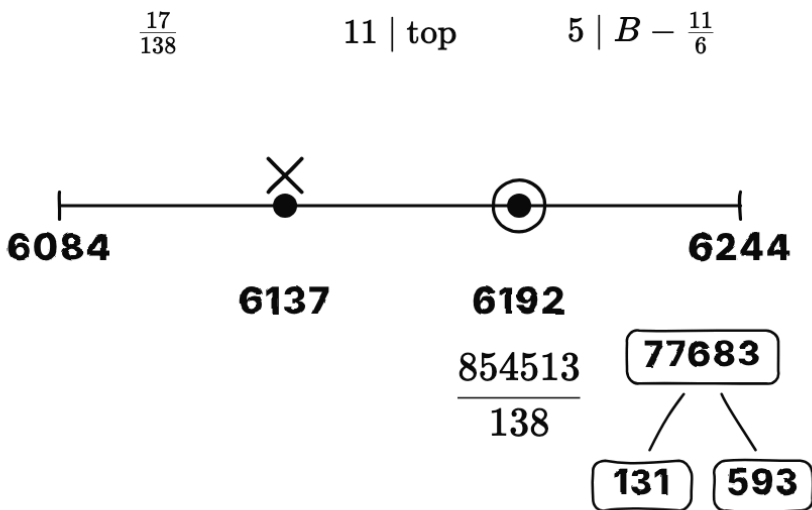
2. Use the fractional-part rule for the index twenty-two: one, minus a half, minus a third, minus a twenty-third.

Hint: Put everything over one hundred and thirty-eight.

Answer: One hundred and thirty-eight, minus sixty-nine, minus forty-six, minus six, all over one hundred and thirty-eight, which is seventeen over one hundred and thirty-eight.

Pinning a Number From Its Size

Think of a guessing game: someone hides a number between six thousand and something and six thousand two hundred and something, and gives you three clues about its digits. With the right clues, a hundred and sixty possibilities shrink to two. Ramanujan plays exactly this game with a Bernoulli number, and then shows where the rough size comes from.



Three clues, two suspects

His example is the Bernoulli number with index twenty-two, and he supposes we only know that it lies between six thousand and eighty-

four and six thousand two hundred and forty-four. The first clue is the fractional part, which the rule of the last part fixes at seventeen over one hundred and thirty-eight. The second is that eleven divides the index, and eleven is missing from the denominator, so it must divide the numerator. The third is the remainder rule: this number minus eleven sixths must have a numerator divisible by five. Only two candidates pass all three: six thousand one hundred and thirty-seven and seventeen over one hundred and thirty-eight, and six thousand one hundred and ninety-two and the same fraction.

To choose between them, Ramanujan uses his guess that the numerator, divided by eleven, is a prime. The first candidate's numerator is also divisible by seven and seventeen, so he rejects it and keeps the second, eight hundred and fifty-four thousand five hundred and thirteen over one hundred and thirty-eight. That is the right value, as his own table shows, but the final step is not sound: that numerator, divided by eleven, is one hundred and thirty-one times five hundred and ninety-three, not a prime either. The method of clues is good; the prime-quotient clue is the weak link.

Where do such rough limits come from? Section eleven quotes a classical formula: a Bernoulli number equals two, times the factorial of its index, divided by two pi raised to the index, times the sum of one over every whole number raised to that index. That last sum is barely more than one for large indices, so the size is governed by the factorial against the power of two pi. For the index two the sum of one over the squares is pi squared over six, a famous result, and the formula gives two times two, over four pi squared, times pi squared over six, which is exactly one sixth. The same sum can be written as a product over all the primes, and the formula can also be written as an

integral, which Ramanujan derives by expanding a fraction into a geometric series.

$$B_n = \frac{2n!}{(2\pi)^n} \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \dots\right)$$

$$\frac{2 \cdot 2}{4\pi^2} \cdot \frac{\pi^2}{6} = \frac{1}{6}$$

$$1 + \frac{1}{2^n} + \dots = \prod_p \frac{1}{1 - p^{-n}}$$

$$\log 10! \approx 15.10441 \qquad \log 3628800 = 15.10441$$

NEEDS MORE DECIMALS

Size from a factorial

Taking logarithms, he uses the well-known expansion of the logarithm of a factorial, whose terms after the first few are themselves built from Bernoulli numbers, with a remainder that is always a fraction of the next term. For ten factorial, its first terms give fifteen point one zero four four one, and the true logarithm of three million six hundred and twenty-eight thousand eight hundred agrees to every digit shown. So the whole part of a Bernoulli number can be estimated from its size, and the fractional part comes from the prime rule; together they give the exact value. The catch, Ramanujan admits, is that for large indices

this needs logarithms to more decimal places than any table provides, and he promises a way around it later.

Remember

Rough limits plus three arithmetic clues, the fractional part, divisibility by a prime of the index and a remainder rule, narrow the Bernoulli number with index twenty-two down to two candidates. Ramanujan's final choice rests on his prime-quotient guess, which is false here, although the value he picks is right. A classical formula gives the size of each Bernoulli number from a factorial, a power of two π and a sum just above one, and through the logarithm of the factorial it yields the whole part, provided very precise logarithms are available.

Questions

1. Which three clues narrow the Bernoulli number with index twenty-two down to two candidates?
 - a. Its fractional part, divisibility of its numerator by eleven, and a remainder rule involving eleven sixths and five
 - b. Its first digit, its last digit and its length
 - c. The value of π , the factorial and the logarithm
 - d. Its denominator, its square and its square root
2. Why is the final step of Ramanujan's choice not sound, even though the value is right?

- a. Because the chosen numerator, divided by eleven, is not a prime either
 - b. Because the limits were wrong
 - c. Because the fractional part is different
 - d. Because seven does not divide the first candidate
3. According to the classical formula, what mainly decides the size of a Bernoulli number?
- a. The factorial of its index against a power of two π
 - b. The number of primes below its index
 - c. Its denominator
 - d. The cotangent of its index

Answers

1. a) The fractional part is fixed at seventeen over one hundred and thirty-eight, eleven must divide the numerator, and the number minus eleven sixths must have a numerator divisible by five.

2. a) He relies on the numerator divided by eleven being prime, but for the chosen value it is one hundred and thirty-one times five hundred and ninety-three.

3. a) The sum in the formula is barely more than one for large indices, so the size is governed by the factorial against the power of two π .

Worked examples

One sixth from the size formula Use the classical formula for the index two, with the sum of one over the squares equal to π^2 over six, to find the first Bernoulli number.

Two times the factorial of two is four.

Two π raised to the second power is four π squared, so the factor is four over four π squared, one over π squared.

Multiplying one over π squared by π squared over six gives one sixth.

Answer: One sixth

Practice

1. Why can rough limits on a Bernoulli number lead to its exact value?

Hint: Think about what the fractional-part rule already tells you.

Answer: The fractional part is known exactly from the primes of the denominator, so only the whole part is unknown. Rough limits leave a range of whole numbers, and arithmetic rules about the numerator rule out all but a few of them; one more fact picks the right one.

2. Ten factorial is three million six hundred and twenty-eight thousand eight hundred. Why is checking its logarithm a good test of the expansion Ramanujan uses?

Hint: Compare the expansion's first terms with the true value.

Answer: The true natural logarithm is about fifteen point one zero four four one, and the first few terms of the expansion already give the same digits, which shows how accurately the expansion estimates sizes; the same expansion, applied to the size formula, estimates the whole part of a Bernoulli number.

A Route to Proof, With One Gap

Try adding one over eleven squared, one over twelve squared, one over thirteen squared, and so on forever. The answer is about nought point nought nine five one six six. There is a shortcut that gets almost all of those digits from three terms: one tenth, minus one over two hundred, plus one sixth divided by a thousand. The one sixth in that shortcut is the first Bernoulli number, and section twelve is about how such shortcuts might prove the patterns Ramanujan observed.

$$\frac{1}{11^2} + \frac{1}{12^2} + \frac{1}{13^2} + \cdots$$

$$\approx 0.0951663$$

$$\frac{1}{10} - \frac{1}{200} + \underbrace{\frac{1}{6}}_{\text{first Bernoulli}} \cdot \frac{1}{1000} = 0.0951667$$

$$\frac{1}{n} - \frac{1}{2n^2} + \frac{B_2}{n^3} - \frac{B_4}{n^5} + \cdots$$

$$\text{left out} \approx 3 \cdot 10^{-7}$$

A three-term shortcut

The shortcut comes from the logarithm of the factorial, whose expansion we met in the last part. Differentiating it twice gives the tail of the sum of reciprocal squares, starting just after a number n , as a series in powers of one over n : one over n , minus one over twice n squared, plus the first Bernoulli number over n cubed, minus the second over n to the fifth, and so on. This series does not converge if you take all of it, so Ramanujan stops after some term and controls the rest. The leftover is always a fraction, between nothing and the whole, of the first term he left out, and he backs this with an integral formula for it. For n equal to ten the first term left out is about three ten-millionths, which is why three terms already gave six correct digits.

Now the main idea. He writes down a long combination: that same tail, a couple of simple terms, and a sum running over the primes, with one term for each prime above three. He claims that the whole combination can be rewritten as a series in powers of one over x whose coefficients are all whole numbers, beginning with one over x to the fifteenth, minus seven over x to the seventeenth, and so on. If that claim is true, comparing coefficients forces certain combinations to be whole numbers: the first Bernoulli number minus one sixth; minus the second, minus one sixth, plus one fifth; and so on, each with one over each prime that belongs to that Bernoulli number.

COMBINATION

$$\longrightarrow \frac{1}{x^{15}} - \frac{7}{x^{17}} + \dots$$

WHOLE NUMBERS?

$$\frac{1}{6} - \frac{1}{6} = 0$$

$$-\frac{1}{30} - \frac{1}{6} + \frac{1}{5} = 0$$

$$\frac{7}{6} - \frac{1}{6} = 1$$

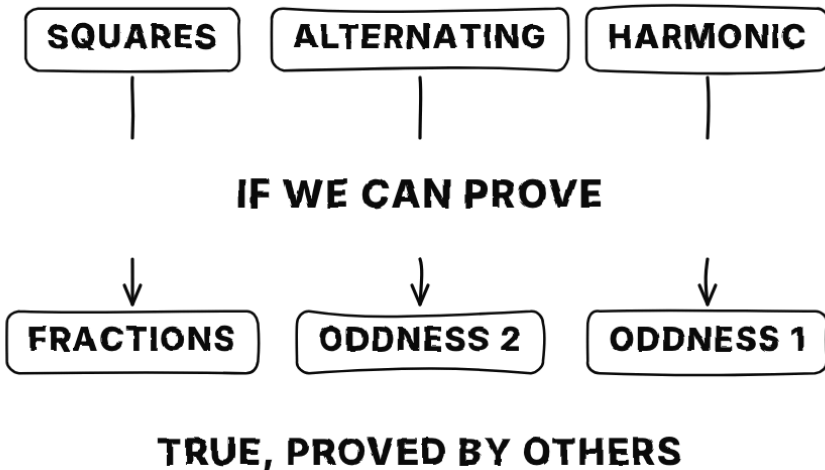
$$-\frac{1}{6} = \frac{1}{2} + \frac{1}{3} - 1$$

The claim and its first checks

Check them with the known values. One sixth minus one sixth is nothing. Minus one thirtieth, minus one sixth, plus one fifth is minus one thirtieth, minus five thirtieths, plus six thirtieths, which is nothing again. For the seventh Bernoulli number, seven sixths, minus one sixth, the result is one, exactly the first coefficient he wrote on the other side. Writing minus one sixth as a half plus a third minus one turns this into the rule for fractional parts from the earlier sections, and the rule about equal denominators follows.

He then repeats the trick twice. Taking the same tail with alternating signs, he gets a series whose coefficients are Bernoulli numbers multiplied by one less than a power of two; if twice that alternating tail had whole-number coefficients, the second half of his oddness

statement would follow. From the first derivative of the logarithm of the factorial, which involves the harmonic sum and Euler's constant, he gets another alternating series; whole-number coefficients there would give the first half of the oddness statement.



Three routes, one missing bridge

Each route ends with the same words: if we can prove. He never proves the whole-number claims, and the editors of the collected papers point this out, adding that some steps use series that do not converge. So section twelve is a careful map of how the patterns could be proved, with the bridge itself still missing; the patterns it aims at are true, and others had proved them earlier.

Remember

The tail of the sum of reciprocal squares has a shortcut series whose coefficients are Bernoulli numbers, and stopping it leaves an error smaller than the first term dropped. Ramanujan reduces the fractional-part rule and the oddness statements to the claim that certain series have whole-number coefficients, and the first few coefficients do come out as whole numbers. He never proves that claim, so this section is a route to proofs, not a proof.

Questions

1. Why can Ramanujan stop the series for the tail after a few terms?
 - a. Because the leftover is always a fraction of the first term he left out
 - b. Because the remaining terms are all zero
 - c. Because the series converges after three terms
 - d. Because the tail is exactly one tenth
2. What claim does the whole proof route rest on?
 - a. That a combination of sums over reciprocal squares and primes has whole-number coefficients
 - b. That every Bernoulli number is prime
 - c. That π is a fraction
 - d. That the factorial grows slowly

3. What happens to the coefficient for the seventh Bernoulli number, seven sixths?
- a. Seven sixths minus one sixth gives one, matching the first coefficient on the other side
 - b. It gives zero
 - c. It gives a fraction
 - d. It cannot be computed

Answers

1. a) The leftover is always a fraction, between nothing and the whole, of the first term he left out.
2. a) He claims the whole combination can be rewritten as a series in powers of one over x whose coefficients are all whole numbers.
3. a) Seven sixths minus one sixth is one, exactly the first coefficient he wrote on the other side.

Worked examples

A coefficient that vanishes Check that minus the second Bernoulli number, minus one sixth, plus one fifth is a whole number, with the second Bernoulli number one thirtieth.

Write everything in thirtieths: minus one thirtieth, minus five thirtieths, plus six thirtieths.

Add: minus one, minus five, plus six is nothing.

So the coefficient is zero, a whole number.

Answer: Zero

Practice

1. Check that the third Bernoulli number, one forty-second, minus one sixth, plus one seventh is a whole number.

Hint: Write everything in forty-seconds.

Answer: One forty-second, minus seven forty-seconds, plus six forty-seconds is nothing, a whole number.

2. In your own words, what is the difference between a route to a proof and a proof?

Hint: Think about the words 'if we can prove'.

Answer: A route shows that the result would follow from some other statement; a proof also establishes that statement. Ramanujan shows his patterns would follow from whole-number coefficients but never proves those, so the patterns are not yet proved by his argument.

Describing a Giant Without Writing It Down

How many digits does two to the power one hundred have? You do not have to multiply: the logarithm of two is about nought point three, so a hundred twos give about thirty point one, which means thirty-one digits. Logarithms measure size without computing, and Ramanujan uses them to describe Bernoulli numbers far beyond his table.

2^{100} has how many digits?

$$100 \times 0.301 = 30.1 \Rightarrow 31 \text{ digits}$$

SIZE WITHOUT COMPUTING

$$\log_{10} B_n \approx \left(n + \frac{1}{2}\right) \log_{10} n - 1.23247 n + 0.7$$

$$\text{true value} \approx \dots + \frac{0.036}{n}$$

$$n = 20 : 20.5 \times 1.301 - 24.65 + 0.7 \approx 2.722$$

$$2.722 + 0.002 \approx 2.724 = \log_{10} 529.12$$

Digits from logarithms

His digit formula says: the logarithm in base ten of a Bernoulli number is about the index plus one half, times the logarithm of the

index, minus one point two three two four seven times the index, plus nought point seven. He gets it from the classical bridge of the last part by taking logarithms and keeping the leading terms of the factorial, and he notes that the true value is a little larger, by about three hundredths divided by the index. Try it for the index twenty: twenty and a half times one point three, minus about twenty-four point six five, plus nought point seven, gives about two point seven two two, and adding the small correction gives two point seven two four, a number of about five hundred and twenty-nine; the tenth value in the table is indeed about five hundred and twenty-nine.

Next he compares neighbours. Dividing the bridge formula for one index by the same formula two places earlier, almost everything cancels, and the ratio of the two Bernoulli numbers is the index times one less than the index, divided by four pi squared, times correction factors that are extremely close to one. So each Bernoulli number is, very nearly, the previous one multiplied by that simple quantity, and the approximation improves rapidly as the index grows. For the index sixteen, the multiplier is two hundred and forty over about thirty-nine point five, roughly six point zero eight, and six point zero eight times seven sixths is about seven point zero nine. So the eighth Bernoulli number lies between seven and eight, and with the prime rule its exact value is six, plus a half, plus a third, plus a fifth, plus a seventeenth, which is three thousand six hundred and seventeen over five hundred and ten. Ramanujan climbs the table this way, rung by rung, up to the tenth number.

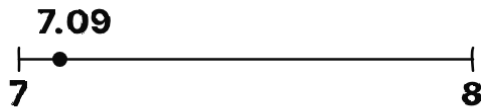
$$B_2 \rightarrow B_4 \rightarrow \cdots \rightarrow B_{20}$$

$$\frac{B_n}{B_{n-2}} = \frac{n(n-1)}{4\pi^2} \left(1 - \frac{2^2-1}{2^n-1}\right) \left(1 - \frac{3^2-1}{3^n-1}\right) \cdots$$

$$B_n \approx \frac{n(n-1)}{4\pi^2} B_{n-2}$$

$$\frac{16 \cdot 15}{4\pi^2} = \frac{240}{39.5} \approx 6.08$$

$$6.08 \times \frac{7}{6} \approx 7.09$$



$$6 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = \frac{3617}{510}$$

Climbing by ratios

Then comes the showpiece: the Bernoulli number with index four hundred and forty-four. Its fractional part follows from the prime rule: a fraction with a denominator of ninety million seven hundred and nine thousand seven hundred and ten. The digit formula gives a logarithm of about six hundred and thirty point two four, so its whole part has six hundred and thirty-one digits, starting with one, seven, five, one. Adding the logarithm of the denominator, the full numerator has six hundred and thirty-nine digits, starting with one, five, eight, eight. And the remainder rule from earlier, divisibility by twenty after subtracting one thirtieth, pins down the end: the whole part ends in four with an even digit before it, and the numerator ends in fifty-seven with an even digit before those. Modern software

confirms all of these digit claims; one step uses his prime-quotient guess, which, as we saw, cannot be trusted in general.

$$B_{444}$$

DENOMINATOR 90709710 $\times \frac{n(n-1)}{4\pi^2}$

WHOLE PART: 631 DIGITS, 1751...

...E4

NUMERATOR: 639 DIGITS, 1588...

...E57

E EVEN

✓ **CHECKED BY COMPUTER**
ONE STEP: PRIME GUESS ?

The giant B444

Remember

Ramanujan's digit formula gives the logarithm of a Bernoulli number from its index alone, and so its number of digits and first digits. Each Bernoulli number is very nearly the previous one times the index, times one less than the index, over four pi squared, which lets him climb the table and fix exact values with the prime rule. For the index four hundred and forty-four he describes the number's length, first digits and last digits without ever writing it out.

Questions

1. What does Ramanujan's digit formula give from the index alone?
 - a. The logarithm of a Bernoulli number, and so its number of digits and first digits
 - b. Its exact numerator
 - c. Whether it is prime
 - d. Its denominator
2. What is the ratio of a Bernoulli number to the one two places earlier, very nearly?
 - a. The index times one less than the index, divided by four pi squared
 - b. Two
 - c. The index divided by pi

- d. One
- 3. Which of Ramanujan's claims about the number with index four hundred and forty-four is not on firm ground?
 - a. The step that uses his prime-quotient guess
 - b. Its number of digits
 - c. Its first digits
 - d. Its fractional part

Answers

- 1. a) His digit formula gives the logarithm in base ten of a Bernoulli number from its index, which tells its number of digits and first digits.
- 2. a) The ratio is the index times one less than the index, divided by four pi squared, times correction factors extremely close to one.
- 3. a) Modern software confirms all the digit claims; one step uses his prime-quotient guess, which cannot be trusted in general.

Worked examples

The eighth Bernoulli number from the seventh Using the neighbour ratio for the index sixteen and the seventh Bernoulli number, seven sixths, find the eighth.

The multiplier is sixteen times fifteen, two hundred and forty, over four pi squared, about thirty-nine point five, which is about six point zero eight.

Six point zero eight times seven sixths is about seven point zero nine, so the number lies between seven and eight.

The prime rule gives the fractional part: a half plus a third plus a fifth plus a seventeenth, minus one, which is forty-seven over five hundred and ten.

So the exact value is seven and forty-seven over five hundred and ten, that is three thousand six hundred and seventeen over five hundred and ten.

Answer: Three thousand six hundred and seventeen over five hundred and ten

Practice

1. How many digits does two to the power one hundred have?

Hint: The base-ten logarithm of two is about nought point three nought one.

Answer: Thirty-one: a hundred times nought point three nought one is about thirty point one, and a number whose logarithm is thirty point something has thirty-one digits.

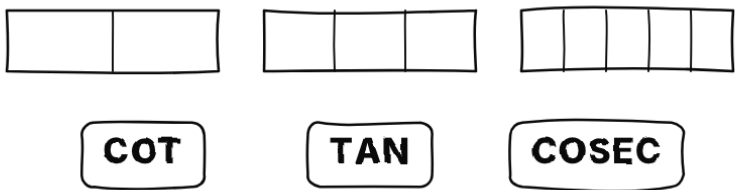
2. The digit formula gives a logarithm of about six hundred and thirty point two four for the number with index four hundred and forty-four. How many digits does its whole part have?

Hint: Add one to the whole part of the logarithm.

Answer: Six hundred and thirty-one digits.

Other Starting Points, and Five-Way Sieves

Imagine a set of sieves for sorting gravel: one splits it in two, a finer one in three, a finer still in five. Ramanujan's method has been working like those sieves on the Bernoulli numbers, and in these sections he widens it in two directions.



POWERS OF 2 MIXED IN

$$\frac{4}{3}B_n(2^n-1) = c_6B_{n-6}(2^{n-6}-1) - \dots + \frac{1}{3}n$$

$$n = 6 : \quad \frac{4}{3} \cdot \frac{1}{42} \cdot 63 = 2 = \frac{1}{3} \cdot 6 \qquad \checkmark$$

$$\left(1 - \frac{1}{2^k}\right) \dots = \dots 3^k$$

$$n = 5 : \quad \frac{5}{6} = \frac{5}{6} \qquad \checkmark$$

Tangent and cosecant

First, the starting function. Everything so far began with the cotangent, but he notes that the tangent and the cosecant, two other functions from school trigonometry, work just as well. They give

relations of the same kind, now with powers of two mixed in. One of them says that four thirds of a Bernoulli number, times one less than two raised to its index, equals an alternating sum of earlier Bernoulli numbers spaced six apart, each with a similar factor, plus a last term that is a third or a sixth of the index. Check it for the index six: four thirds, times one forty-second, times sixty-three, is two; and the last term, a third of six, is also two. A second relation uses factors of one minus a power of a half, and powers of three on the other side; for the index five both sides come to five sixths.

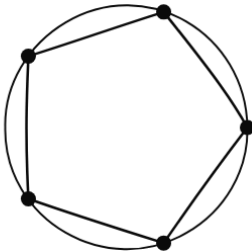
Second, the number of ways to cancel. Ramanujan names his earlier results the one-interval, two-interval and three-interval formulas, after how far apart the surviving Bernoulli numbers sit. Then he states the general rule: take the numbers whose p -th power is one, or whose p -th power is the imaginary unit when p is even, mix them into the cotangent, and you get p -interval formulas. Each such formula keeps only one strand of Bernoulli numbers, spaced twice p apart.

WAYS TO CANCEL

1 INTERVAL 2 INTERVAL 3 INTERVAL

$$\omega^p = 1 \text{ (or } i) \rightarrow \cot \rightarrow p\text{-interval}$$

spacing $2p$



$$\sum_{k=0}^4 \sin x \alpha^k$$

$$= 5 \left(\frac{x^5}{5!} - \frac{x^{15}}{15!} + \dots \right)$$

EVERY 10TH POWER

Five roots on a pentagon

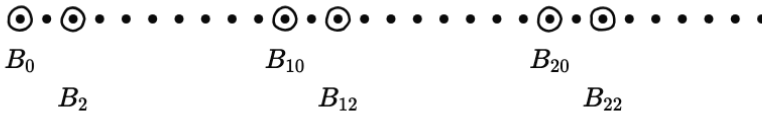
He works out the case of five. The five fifth roots of one sit at the corners of a regular pentagon on a circle. Adding the sines of x times each of them, everything cancels except every tenth power: the sum is five times x to the fifth over five factorial, minus x to the fifteenth over fifteen factorial, and so on. A product of five sines then turns into a combination of this sum at three special arguments, involving one plus or minus the square root of five. The square root of five brings in the golden ratio, about one point six one eight, and with it a sequence of whole numbers made from its powers, one, three, four, seven, eleven, each the sum of the two before, close cousins of the Fibonacci numbers.

$$1 \pm \sqrt{5}$$

$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

$$1, 3, 4, 7, 11, \dots$$

$$3 + 4 = 7, \quad 4 + 7 = 11$$



$$-x \cdot \frac{x^4/24}{x^5/120} = -5 \quad \checkmark$$

COSINES: MORE

TAN, COSEC TOO

spacing $2p$

Golden ratio strands

With these whole numbers he writes five new formulas, each a ratio of two series that isolates one strand: the zeroth, tenth, twentieth Bernoulli numbers by index; then the second, twelfth, twenty-second; and so on. You can test the first one: for a tiny x its right side is minus x , times x to the fourth over twenty-four, divided by x to the fifth over a hundred and twenty, which is minus five; the left side begins with five times the zeroth Bernoulli number, and since that number is minus one, five times it is minus five, so the two sides agree. He adds that a similar identity built from the product of five cosines gives many more such identities.

Remember

The tangent and the cosecant can replace the cotangent and give further relations among Bernoulli numbers, which check out for small indices. Mixing the p -th roots of one into the cotangent gives p -interval formulas, each keeping a strand of Bernoulli numbers spaced twice p apart. In the five-interval case the pentagon brings in the golden ratio and the whole numbers one, three, four, seven, eleven, and each formula isolates every tenth Bernoulli number by index.

Questions

1. Which functions can replace the cotangent as a starting point?
 - a. The tangent and the cosecant
 - b. The logarithm and the exponential
 - c. The square root
 - d. Only the sine
2. What does a p -interval formula keep?
 - a. One strand of Bernoulli numbers spaced twice p apart
 - b. All Bernoulli numbers
 - c. Only the first p Bernoulli numbers
 - d. Only prime-index Bernoulli numbers
3. Where does the golden ratio come from in the five-interval case?

- a. From the square root of five that appears with the fifth roots of one
- b. From the number π
- c. From the factorial
- d. From the denominators

Answers

- 1. a) He notes that the tangent and the cosecant work just as well as the cotangent.
- 2. a) Each such formula keeps only one strand of Bernoulli numbers, spaced twice p apart.
- 3. a) The square root of five brings in the golden ratio, about one point six one eight.

Worked examples

The tangent relation for the index six Check the first tangent relation for the index six: four thirds of the third Bernoulli number, one forty-second, times one less than two to the sixth, should equal the last term, a third of six.

Two to the sixth is sixty-four, and one less is sixty-three.

Four thirds times one forty-second times sixty-three: sixty-three forty-seconds is three halves, and four thirds of three halves is two.

A third of six is also two, so both sides agree.

Answer: Both sides equal two

Practice

1. Continue the whole numbers one, three, four, seven, eleven: what are the next two?

Hint: Each is the sum of the two before.

Answer: Eighteen and twenty-nine: seven plus eleven is eighteen, and eleven plus eighteen is twenty-nine.

2. In the five-interval case, which Bernoulli numbers does the first formula keep, by index?

Hint: The strands are spaced twice five apart.

Answer: The ones with index zero, ten, twenty, thirty and so on.

The Four-Way Sieve and the Final Verdict

The five-way sieve of the last part ran on the golden ratio; the four-way sieve runs on the square root of two. Ramanujan builds two helper sequences from the numbers one plus one over the square root of two, and one minus it. The first adds their powers, the second subtracts them. Adding cancels the irrational parts: for the second power, the two squares add up to three, and for the third power to five. The sequences obey simple building rules, so each value comes from earlier ones, much as the golden-ratio numbers did.

$$1 \pm \frac{1}{\sqrt{2}}$$

$$a_n = \left(1 + \frac{1}{\sqrt{2}}\right)^n + \left(1 - \frac{1}{\sqrt{2}}\right)^n$$

$$b_n = \left(1 + \frac{1}{\sqrt{2}}\right)^n - \left(1 - \frac{1}{\sqrt{2}}\right)^n$$

$$a_2 = 3, \qquad a_3 = 5$$

$$a_{m+n} = a_m a_n - a_{m-n} / 2^n$$

$$B_0, B_8, B_{16}, \dots \qquad B_2, B_{10}, B_{18}, \dots$$

$$B_4, B_{12}, B_{20}, \dots \qquad B_6, B_{14}, B_{22}, \dots$$

$$-x \cdot \frac{x^3/6}{x^4/24} = -4 = 4B_0 \quad \checkmark$$

SIEVES OF 2, 3, 4, 5

The four-way sieve

With them he writes four formulas, each a ratio of two series that keeps one strand of Bernoulli numbers spaced eight apart by index: the zeroth, eighth, sixteenth; then the second, tenth, eighteenth; then the fourth, twelfth, twentieth; and finally the sixth, fourteenth, twenty-second. The first is easy to test: for a tiny x the right side is minus x , times x cubed over six, divided by x to the fourth over twenty-four, which is minus four, and the left side begins with four times the zeroth Bernoulli number, minus one, which is also minus four. With that the paper ends: one idea, cancelling terms with roots of one or of the imaginary unit, has produced sieves of two, three, four and five.

The book then prints a note from the editors of the collected papers, and it is the right place to take stock. First, small clarifications: Ramanujan numbers his Bernoulli numbers differently from most books, and the counting numbers in his equations are the usual binomial coefficients. His fast recurrence holds for odd indices above one, with three forms of its last term.

EDITORS

OTHER NUMBERING

c_k binomial

(12): odd $n > 1$

TWO DERIVATIONS INVALID ✕

KNOWN: VON STAUDT–CLAUSEN, EULER

IF PROVED	FOLLOWS	CONJECTURE
●	●	●
● ?	●	●
●		●

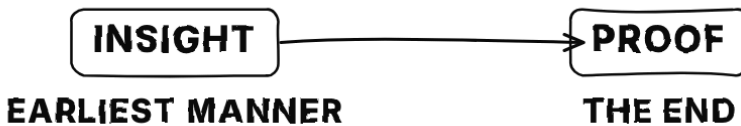
TELL THEM APART

Three kinds of claim

Then the verdict. Two of his derivations, the ones that turned the size formula and the tail of the reciprocal squares into integrals, are invalid as they stand, because a series inside the integral stops converging. His main arithmetic results were already known: the rule for fractional parts is the theorem of von Staudt and Clausen, and the oddness statement follows from it and was known to Euler. Of his eight arithmetic statements, he indicated proofs for three, the oddness statement, the fractional-part rule and the rule about equal denominators; but those proofs depend on his claims that certain series have whole-number coefficients, and those claims are never proved. Two more, the power-of-two statement and the prime-quotient statement, are presented as consequences of others. And three, the prime numerator, the factors two and three in every

denominator, and the denominator rule, are stated merely as conjectures.

So three kinds of claim run through this paper: conjectures read off a table, consequences that are only as safe as what they rest on, and conditional proofs that wait for one unproved step. Telling these apart is the most useful skill this paper can teach. In the editors' words, the paper is interesting as a specimen of Ramanujan's earliest manner, before he learned to support his insight with proof. This course has followed his first paper from the cotangent to that verdict, and here it ends.



STRANDS SPACED 8, FROM $1 \pm 1/\sqrt{2}$

RESULTS KNOWN, PROOFS CONDITIONAL

3 CONDITIONAL, 2 FOLLOW, 3 GUESSES

From insight to proof

Remember

The four-interval formulas use helper sequences built from one plus or minus one over the square root of two, and each keeps a strand of Bernoulli numbers spaced eight apart; the first checks out against the zeroth number. The editors note that some derivations are invalid as written, that the main results were known to von Staudt, Clausen and Euler, and that Ramanujan's indicated proofs rest on unproved claims about whole-number coefficients. Of his eight arithmetic statements, three are conjectures, two are consequences and three have only conditional proofs, a sorting worth keeping in mind whenever you read mathematics.

Questions

1. What do the four-interval formulas each keep?
 - a. A strand of Bernoulli numbers spaced eight apart by index
 - b. All the Bernoulli numbers
 - c. Only the first four Bernoulli numbers
 - d. Every second Bernoulli number
2. Why do Ramanujan's indicated proofs not count as complete?
 - a. They depend on claims that certain series have whole-number coefficients, which are never proved
 - b. They use the wrong Bernoulli numbers
 - c. They are too long

- d. They were never written down
3. Which three statements does the editors' note list as mere conjectures?
- a. The prime numerator, the factors two and three in every denominator, and the denominator rule
 - b. The oddness statement, the fractional-part rule and the equal-denominator rule
 - c. The power-of-two statement and the prime-quotient statement
 - d. All eight

Answers

1. a) Each is a ratio of two series that keeps one strand of Bernoulli numbers spaced eight apart by index.
2. a) Those proofs depend on his claims that certain series have whole-number coefficients, and those claims are never proved.
3. a) Three, the prime numerator, the factors two and three in every denominator, and the denominator rule, are stated merely as conjectures.

Worked examples

Testing the first four-interval formula For a tiny x , compare the right side of the first four-interval formula, minus x times x cubed over six divided by x to the fourth over twenty-four, with the left side, four times the zeroth Bernoulli number.

x cubed over six divided by x to the fourth over twenty-four is twenty-four over six, divided by x , which is four over x .

Multiplying by minus x gives minus four.

The zeroth Bernoulli number is minus one, so four times it is also minus four.

The two sides agree.

Answer: Both sides are minus four

Practice

1. Add the squares of one plus one over the square root of two and one minus one over the square root of two.

Hint: Each square is one, plus or minus the square root of two, plus one half.

Answer: Three: the plus and minus square roots of two cancel, leaving twice one and a half.

2. In your own words, describe the three kinds of claim the editors distinguish, and why the distinction matters.

Hint: Think of a guess, a consequence and an argument with a missing step.

Answer: A conjecture is a guess from examples; a consequence is only as safe as the statement it follows from; a conditional proof is complete except for one unproved step. Keeping them apart tells you how much to trust each result, which matters because some of Ramanujan's conjectures are true and one is false.