

Collected Papers of Srinivasa Ramanujan



Srinivasa Ramanujan · 1927

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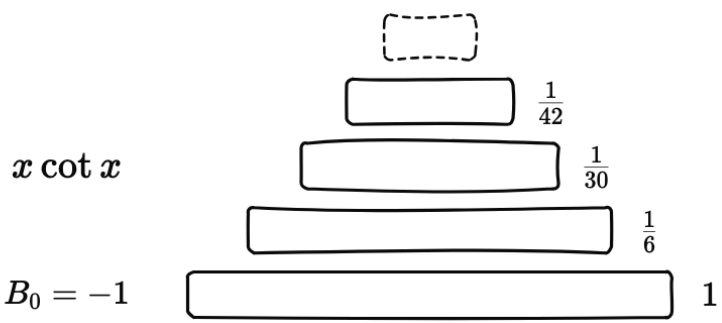
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The Recipe Inside the Cotangent

Imagine finding a recipe card where every ingredient amount is a strange fraction, and you suspect the fractions follow a rule nobody wrote down. That is roughly where Ramanujan stood in 1911, as a young clerk in Madras, when he sent his first paper to the Journal of the Indian Mathematical Society. The strange fractions are the Bernoulli numbers: one sixth, one thirtieth, one forty-second, then larger and larger ones. They turn up wherever mathematicians add up powers, estimate factorials, or study the prime numbers.



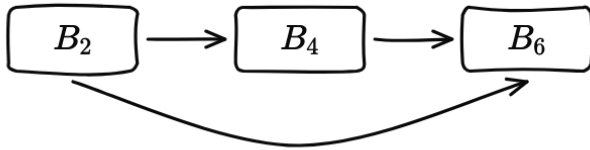
A dish built from layers

Ramanujan meets them in a familiar place: the cotangent from school trigonometry, multiplied by the angle. Close to zero, this function can be built up as a long sum of simpler and simpler pieces, like a dish assembled from layers. The amount of each layer is set by one Bernoulli number. In Ramanujan's way of writing them they are all positive, and to keep the first layer in line with the rest he agrees to call the zeroth one minus one. So the question of the paper is a cook's question: if the dish is known, how do you read off the amounts of each ingredient?

His first answer is to write the same dish in three different ways, using three school identities for the cotangent. When two descriptions of the same thing are equal, every layer must match the corresponding layer on the other side. Matching them one at a time gives a chain of equations, and solving the chain hands out the Bernoulli numbers in order: one sixth first, then one thirtieth, then one forty-second. It works, but it is like a recipe in which each new step needs every earlier step re-weighed: the further you go, the more you must carry along, all of the earlier numbers at every step. Ramanujan, who did his calculations by hand on a slate, says plainly that this becomes tedious, and sets out to find shortcuts.

$$(x \cot x)^2$$

$$B_2 B_2, \quad B_2 B_4, \quad \dots$$



Match layer by layer

The first shortcut multiplies the dish by itself. There is an exact rule saying what the square of this function equals, and comparing the layers of the square gives a new chain, where each Bernoulli number comes from products of earlier pairs. It reaches the same values by another road, a little like checking a sum by adding the column upwards instead of downwards.

The second shortcut is more cunning. The cotangent has a twin, the hyperbolic cotangent, which you get by feeding the same function an imaginary angle. When you add the two, every second layer cancels out, the way two waves can cancel each other; when you subtract them, the other half cancels. So each half of the family can be studied on its own, and the equations that come out need only half as many

earlier numbers as before. The same trick also works with the tangent and its hyperbolic twin.

COT

COTH

SUM

BEFORE

NOW

TAN + TANH

Twins that cancel

This is the habit that runs through the whole paper: find a clever combination that silences most of the terms, so that the remaining ones can be found with less work. A calculator designer does exactly the same today, choosing formulas that need the fewest operations to give the right digits.

Remember

The Bernoulli numbers are the hidden amounts that build up the cotangent multiplied by the angle; Ramanujan takes them all positive and calls the zeroth one minus one. Writing the same function in several ways and matching its layers produces them in order, but each new one needs all of the earlier ones. Squaring the function, or pairing it with its hyperbolic twin so that every second layer cancels, gives faster chains that need only half as many earlier numbers.

Questions

1. Why does Ramanujan agree to call the zeroth Bernoulli number minus one?
 - a. Because all his Bernoulli numbers are negative
 - b. To keep the first layer of the cotangent series in line with the rest
 - c. Because the cotangent is negative near zero
 - d. To make the chain of equations shorter
2. Why does writing the same function in different ways give equations?
 - a. Because two equal descriptions must match layer by layer
 - b. Because each description gives a different answer to average
 - c. Because only one description is correct
 - d. Because the cotangent is a polynomial

3. What is wrong with the first chain of equations?
 - a. It gives wrong values
 - b. Each new step needs all the earlier numbers, so the work keeps growing
 - c. It only works for the first three numbers
 - d. It needs an imaginary angle
4. What does pairing the cotangent with its hyperbolic twin achieve?
 - a. Every second layer cancels, so each half of the family needs only half as many earlier numbers
 - b. It makes all the layers equal
 - c. It removes the need for any earlier numbers
 - d. It turns the fractions into whole numbers

Answers

1. **b)** In his way of writing them they are all positive, and calling the zeroth one minus one keeps the first layer in line with the rest.
2. **a)** When two descriptions of the same thing are equal, every layer must match the corresponding layer on the other side, and each match is an equation.
3. **b)** Each new step needs every earlier step re-weighted, so the further you go, the more you must carry along.
4. **a)** Adding the two cancels every second layer and subtracting cancels the other half, so each half can be studied on its own with half as many earlier numbers.

Practice

1. Explain in your own words why cancelling half of the terms makes the Bernoulli numbers quicker to find.

Hint: Think about how many earlier numbers each new step has to carry along.

Answer: In the first chain every new number depends on all the earlier ones, so the work grows step by step. When half of the terms cancel, each half of the family depends only on its own members, so each new number needs only half as many earlier ones, and the work for each step is roughly halved.

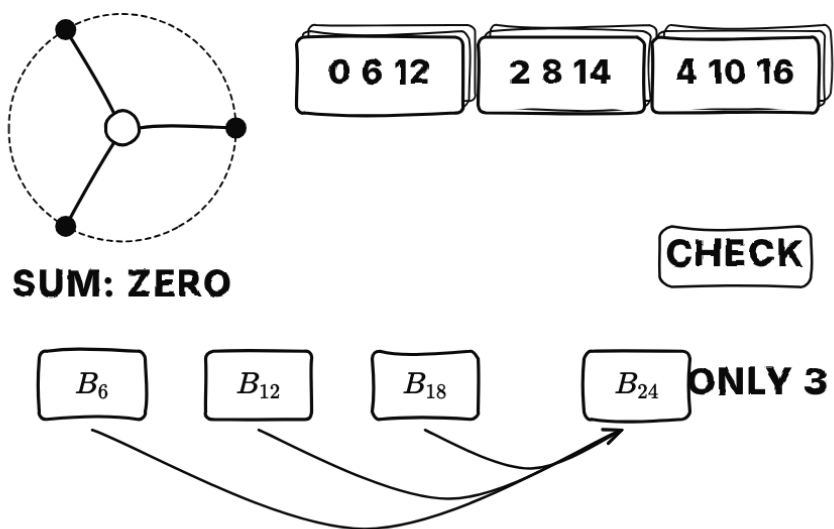
2. Why is it useful to have a second, different road to the same values, such as the squared function?

Hint: Think of how an accountant checks a long sum.

Answer: A second road that reaches the same values serves as a check: if both routes give the same number, an arithmetic slip is unlikely, just as adding a column upwards and downwards should give the same total.

Three Ropes and a Table of Specimens

Picture a ring with three ropes tied to it, a third of a turn apart, and three friends pulling on them equally hard. The ring does not move: the three pulls cancel each other perfectly. Ramanujan's next shortcut uses exactly this kind of balance, with numbers instead of ropes.



Three ropes, three piles

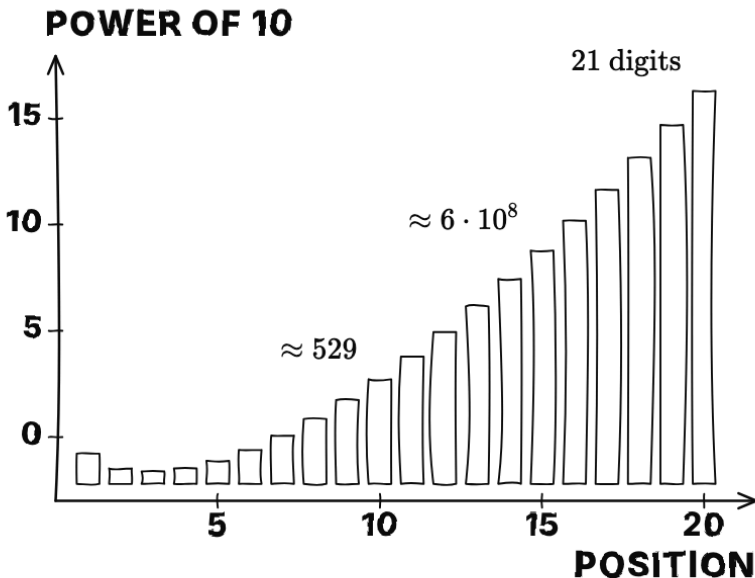
In the previous part, pairing the cotangent with its hyperbolic twin made every second term cancel. Now he uses three partners instead of two. They come from the cube roots of one: the number one and two complex numbers sitting a third of a turn away from it on a circle,

which, like the three ropes, add up to nothing. When he feeds all three into the cotangent and adds the results, two terms out of every three cancel. What survives is a thin strand of Bernoulli numbers, every sixth one, and by shifting the mixture he gets the other two strands as well. It is like sorting a deck of cards into three piles so that each pile can be counted on its own.

Each strand comes with its own simple description, and from these descriptions he gets a new chain of equations. In this chain each new Bernoulli number needs only a third of the earlier numbers that the very first method needed, so the hand work shrinks a great deal. To get the one in the twenty-fourth place, for instance, he needs only three earlier ones.

He also gets a second, quite different relation, in which each number is built from products of pairs of earlier ones. He does not use it to compute; he keeps it as a check. This is the same habit as an accountant who adds a column twice by different routes, or a programmer who tests a fast routine against a slow but reliable one. If both roads lead to the same fraction, a slip in the arithmetic is very unlikely.

With these tools ready, Ramanujan turns from method to observation. He explains that he wants to discover the properties of these numbers, and that for this it is convenient to have a good number of them in front of him. So he computes them in turn and lists the first twenty, a table of specimens like the ones a naturalist pins in a case before looking for a pattern.



Gentle, then explosive

The table tells a vivid story. The first values are gentle fractions: one sixth, one thirtieth, one forty-second, and one thirtieth again. Then they start to climb, slowly at first and then explosively: the tenth is already above five hundred, the fifteenth is around six hundred million, and the twentieth has a numerator twenty-one digits long. Meanwhile the bottoms of the fractions stay quite small, in the thousands or tens of thousands, and Ramanujan notes that the sequence grows without any limit.

Today a computer algebra system prints this table in a blink. In 1911 every entry was hand work, which is why a recipe needing a third of the earlier numbers, plus an independent check, made the difference between a table full of errors and a reliable one. And this table is what

Ramanujan studied next, looking for hidden rules in the tops and bottoms of these fractions.

Remember

The three cube roots of one add up to nothing, like three equal pulls on a ring, and mixing them into the cotangent cancels two terms in every three. That sorts the Bernoulli numbers into three strands, every sixth one, and gives a chain that needs only a third of the earlier numbers. A second relation, built from products of pairs, serves as an independent check on the arithmetic. With these tools Ramanujan computes a table of twenty Bernoulli numbers, gentle fractions at first, then enormous numbers with modest denominators, the raw material for his search for patterns.

Questions

1. What happens when the three cube roots of one are fed into the cotangent and the results are added?
 - a. Two terms out of every three cancel, leaving one strand
 - b. All the terms double
 - c. The series becomes finite
 - d. Every second term cancels
2. Why does Ramanujan keep a second relation, built from products of pairs, alongside the fast chain?

- a. Because it is faster
 - b. As a check: two different roads to the same fraction make a slip very unlikely
 - c. Because the fast chain gives wrong answers
 - d. To find the denominators
3. What does the table of twenty values show?
- a. The values shrink steadily
 - b. Gentle fractions at first, then explosive growth, while the denominators stay small
 - c. Only whole numbers
 - d. Values that repeat every six places

Answers

1. **a)** Like three equal pulls on a ring, the three numbers balance out, and two terms out of every three cancel.
2. **b)** He does not use it to compute; he keeps it as a check, like adding a column twice by different routes.
3. **b)** The first values are gentle fractions, then they climb explosively, while the bottoms of the fractions stay quite small.

Practice

1. Explain with your own analogy why splitting the Bernoulli numbers into three strands saves work.

Hint: Think of sorting something into piles before counting.

Answer: Sorting a big pile of coins into three smaller piles lets each pile be counted on its own; in the same way, each strand of Bernoulli numbers

can be computed from its own few earlier members, so each new number needs only about a third of the earlier ones.

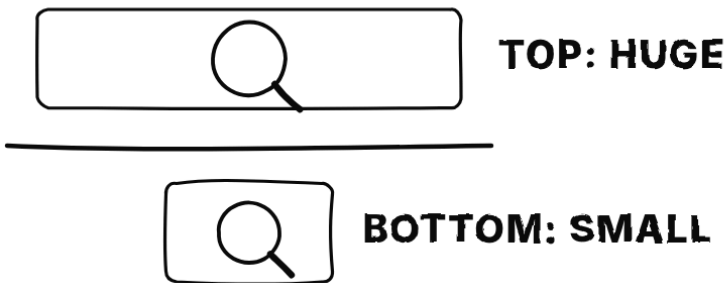
2. Why did the speed of a recipe matter more to Ramanujan in 1911 than it would today?

Hint: Think about how the numbers were computed then and now.

Answer: Every entry was worked out by hand, and the numbers quickly grow to many digits, so each extra term meant long, error-prone arithmetic. Today a computer algebra system prints the whole table instantly.

Guessing the Rules From the Table

Imagine you have collected twenty seashells and you start sorting them: the ridged ones here, the spiral ones there, and you begin to suspect a rule about which shapes go with which colours. That is the mood of the next stretch of the paper. Ramanujan has his table of twenty Bernoulli numbers, and he starts stating what he sees in them. He is careful to say "it will be observed", and he promises reasons only later, so these are guesses read off the specimens, not proofs.



Two halves of a fraction

Each Bernoulli number is a fraction with a top and a bottom, and the two halves behave very differently. The bottoms are small, tidy numbers, while the tops grow huge and look random. So Ramanujan studies the two halves separately, the way a detective studies the footprints and the fingerprints apart before putting the story together.

For the bottoms he finds a rule so simple it could be a pencil game. You take the index of the Bernoulli number, list everything that divides it, add one to each, keep only the primes, and multiply them: that is the bottom of the fraction. It matches every entry in his table. He also notices that the leftover piece of each number, after removing its whole part, is built from those same primes, and that when two Bernoulli numbers share the same bottom, either their difference or their sum, depending on the pair, is always a whole number. These rules about the bottoms turn out to be completely true: they are a theorem that two mathematicians, Clausen and von Staudt, had found some seventy years earlier, and Ramanujan rediscovered it alone.

INDEX 24

	1	2	3	4	6	8	12	24
+1 ↓	2	3	~~4~~	5	7	~~9~~	13	~~25~~

$$2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 = 2730$$

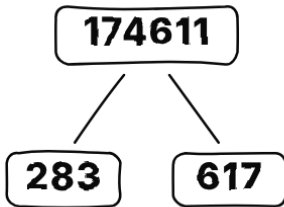


The pencil game

For the tops he is more daring. He guesses that, after a small adjustment, the top is always a prime number, and that a prime dividing the index must show up either in the top or in the bottom, never in neither. He also sees that certain combinations of a Bernoulli number with powers of two always produce odd whole numbers, and that some produce numbers that leave a remainder of one when divided by thirty.

TOP: PRIME?

ODD NUMBERS



✓
EULER

CONJECTURES

A bold guess breaks

Here the story needs a warning label. The oddness pattern is true and was already known to Euler, but the guess that the top is always prime is simply false; it breaks at the tenth number of his own table. The editors who collected his papers after his death say this plainly: several of these statements were stated only as conjectures, and the proofs he sketches later rest on steps he never proved.

This is not a failure; it is how mathematics often moves. Someone looks hard at data and writes down bold guesses; then others test them, and each guess ends up as a theorem or a lesson. Modern researchers do the same with computers, generating huge tables of numbers and searching for patterns, which experts then try to prove.

Ramanujan simply did it by hand, at twenty-three, with remarkable instinct for which patterns were real.

Remember

From his table, Ramanujan studies the tops and the bottoms of the Bernoulli numbers separately and states what he observes. His rule for the bottoms, built from the primes that sit one above the divisors of the index, is completely true, a known theorem he rediscovered. His guesses about the tops are bolder: some, like the oddness pattern, are true, but the claim that the top is always prime fails, and his editors list several of these statements as conjectures, not proved facts.

Questions

1. Why are the statements in this stretch of the paper guesses rather than proofs?
 - a. Because Ramanujan says they are observed from the table and promises reasons only later
 - b. Because the editors removed the proofs
 - c. Because they are copied from another book
 - d. Because they concern only the first number
2. What happened to his rule for the bottoms of the Bernoulli numbers?
 - a. It turned out to be false

- b. It turned out to be completely true, a theorem Clausen and von Staudt had found earlier
 - c. It was never checked
 - d. It only works for the first five numbers
3. Which of his guesses about the tops is false?
- a. That certain combinations are odd whole numbers
 - b. That the top is always a prime after a small adjustment
 - c. That primes in the index show up in the top or the bottom
 - d. That the numbers grow

Answers

1. **a)** He is careful to say 'it will be observed', and he promises reasons only later, so these are guesses read off the specimens.
2. **b)** These rules about the bottoms are completely true: a theorem Clausen and von Staudt had found some seventy years earlier, which Ramanujan rediscovered alone.
3. **b)** The oddness pattern is true and was known to Euler, but the guess that the top is always prime breaks at the tenth number of his own table.

Practice

1. In your own words, why is it useful to study the tops and the bottoms of the fractions separately?

Hint: Think about how differently the two halves behave in the table.

Answer: The bottoms are small and tidy while the tops are huge and look random, so they probably follow different kinds of rules. Looking at each half on its own makes the simple rule for the bottoms visible, instead of hiding it inside fractions that look messy as a whole.

2. Explain why a false guess, like the prime-top rule, can still be valuable in mathematics.

Hint: Think about what happens after a guess is written down.

Answer: A stated guess can be tested. If it fails, the counterexample shows exactly where intuition went wrong and stops others from building on it; if it holds, someone proves it. Either way the guess moves the subject forward, as happened with Ramanujan's table.

Detective Work, and a Proof Left Hanging

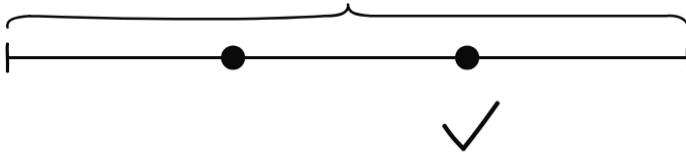
Imagine a detective who knows only that the suspect is between thirty and forty, left-handed and owns a red car. Any one clue leaves many people; together they leave two or three. In this stretch of the paper Ramanujan becomes that detective, and his suspect is a Bernoulli number.

FRACTION

PRIME | TOP

REMAINDER

160 NUMBERS



WITNESS UNRELIABLE

The detective's two suspects

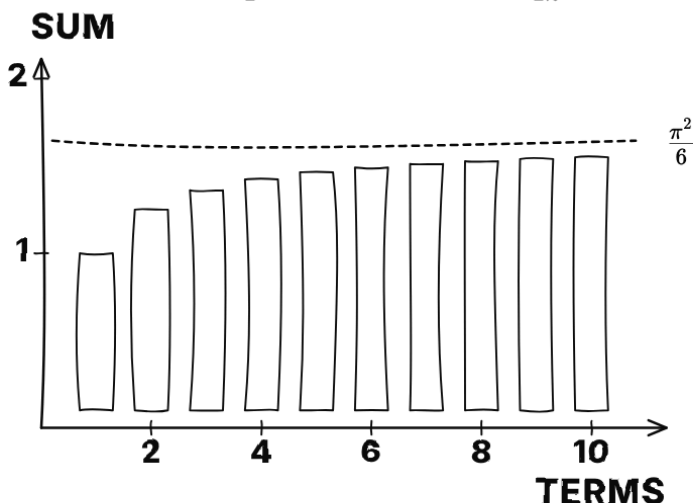
His case is the Bernoulli number in the eleventh place of his table. He pretends to know only rough limits on its size, a range of a hundred

and sixty whole numbers. Then he brings his clues. The leftover fraction after the whole part is fixed by the primes of the denominator, so that part is known exactly. A prime that divides the index but is missing from the denominator must divide the top of the fraction. And a rule about remainders when dividing by five rules out most of what is left. Only two candidates survive.

To choose between them, he uses one more clue: his guess that the top, after dividing out that prime, is itself a prime. One candidate fails that test, so he keeps the other, and it is indeed the value in his table. But the clue he used to decide is not trustworthy: for the number he chose, the top is not a prime after division either. He reached the right suspect, partly by luck, with one unreliable witness.

Where do the rough limits come from in the first place? Here he borrows a classical bridge. There is a famous sum: add one over every whole number squared, or cubed, or raised to any even power, and you get a definite number; for squares it is π^2 divided by six. The bridge says that each Bernoulli number equals a factorial divided by a power of two π , multiplied by one of these sums. For large indices the sum is barely more than one, so the size of a Bernoulli number is essentially a factorial against a power of two π , which explains why the table explodes. The same sum can also be written as a product with one factor for every prime, a hint that primes are woven into these numbers from the start.

$$\log n! \approx \left(n + \frac{1}{2}\right) \log n - n + \cdots + \frac{B_2}{2n} - \cdots$$

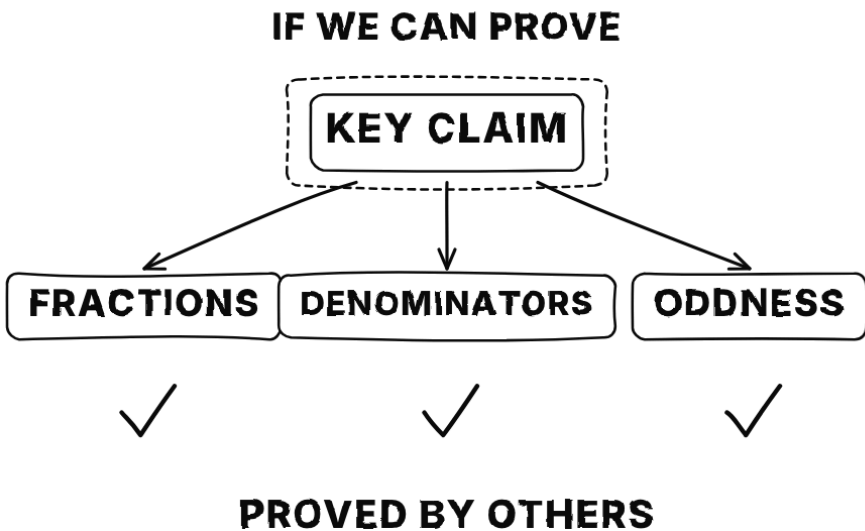


A sum that settles

Ramanujan also turns the bridge into integral formulas, areas under curves, which let the size be computed in other ways. And he plugs it into a well-known recipe for the logarithm of a factorial, whose correction terms are again Bernoulli numbers. With such a recipe, the whole part of any Bernoulli number can be estimated, and the leftover fraction comes from the prime rule, so the exact value follows. The catch, which he states openly, is that this needs logarithms to more decimal places than the printed tables of his day offered, and he promises a way around it later.

Then he turns to proving his own patterns, and the strategy is elegant. He gathers several infinite sums, built from the same reciprocal squares and from one term for each prime, and claims that a certain

combination of them can be rewritten using only whole numbers as coefficients. If that single claim were true, his rules about leftover fractions and equal denominators would follow at once, and two similar claims would give his oddness patterns. It is like reducing a whole family of court cases to one key piece of evidence: prove that, and everything else follows. The first few coefficients he can check by hand do come out as whole numbers, which is encouraging.



One key piece of evidence

But the key piece of evidence is never produced. Each argument ends with the words "if we can prove", and he never does. The editors of his collected papers point out that these claims remain unproved and that some steps use sums that do not settle down to a value, so the arguments are sketches. The rules he was aiming at are true all the

same: they had been proved earlier by Clausen and von Staudt, and by Euler.

For a learner, this is a valuable lesson in reading mathematics. A correct answer is not the same as a correct argument, and a promising plan is not yet a proof. Ramanujan, at twenty-three, was already superb at finding the right statements; rigorous proof was something he would develop later, working with Hardy in Cambridge.

Remember

Rough limits on a Bernoulli number, combined with exact clues about its leftover fraction and the primes in its top, narrow the candidates to two; Ramanujan picks the right one, but with a clue that is false. A classical bridge ties each Bernoulli number to a factorial, a power of two π and a sum over all whole numbers, which explains their explosive growth and gives their size from precise logarithms. His proof plan reduces his patterns to one claim about whole-number coefficients, a claim he never proves, though the patterns themselves are true theorems found earlier by others.

Questions

1. Why do several clues together narrow the candidates far more than any single clue?
 - a. Each clue rules out most of what the others leave, so only two candidates survive

- b. Because one clue is always enough
 - c. Because the clues repeat each other
 - d. Because the range was tiny to begin with
2. What is wrong with the final clue Ramanujan uses to choose between the two candidates?
- a. It relies on his guess that the top divided by the prime is prime, which is false for the chosen number
 - b. It picks the wrong number
 - c. It ignores the leftover fraction
 - d. It needs a calculator
3. Why do Bernoulli numbers grow so explosively, according to the classical bridge?
- a. Each is essentially a factorial against a power of two π
 - b. Their denominators shrink
 - c. They double at each step
 - d. They are squares of primes
4. What is the status of Ramanujan's proof plan?
- a. It reduces everything to one claim about whole-number coefficients that he never proves
 - b. It is a complete proof
 - c. It was disproved
 - d. It proves the prime-top guess

Answers

1. a) Like a detective's clues, any one leaves many possibilities, but together the leftover fraction, the prime in the top and the remainder rule leave only

two.

2. a) The clue is not trustworthy: for the number he chose, the top is not a prime after division either, although the number is right.

3. a) For large indices the sum is barely more than one, so the size is essentially a factorial against a power of two pi.

4. a) Each argument ends with 'if we can prove', and the key claim is never produced; the editors point this out.

Practice

1. Explain in your own words why a correct answer does not guarantee a correct argument, using Ramanujan's choice of the eleventh Bernoulli number.

Hint: Think about the clue he used to decide.

Answer: He picked the right value, but the reason he gave, that the top divided by eleven is prime, is false for that very number. So the answer was right partly by luck; an argument is only correct if every step in it is true.

2. Why is it useful that the sum over all whole numbers can also be written with one factor for every prime?

Hint: Think about what the Bernoulli numbers seem to know about primes.

Answer: It shows that primes are built into the Bernoulli numbers from the start, which helps explain why their denominators and fractional parts are governed by primes, and it links these numbers to the study of prime numbers.

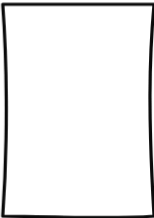
Measuring Giants From Afar

Astronomers can tell you how far away a star is and how bright it really shines without ever travelling there. Ramanujan does something similar with Bernoulli numbers that are far too large to write out: he measures them from a distance.

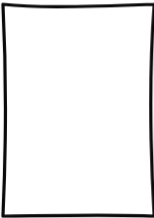
$$B_n \approx \frac{2 n!}{(2\pi)^n}$$

LOG → NUMBER OF DIGITS

TABLE: 529.12



RECIPE: 529.12



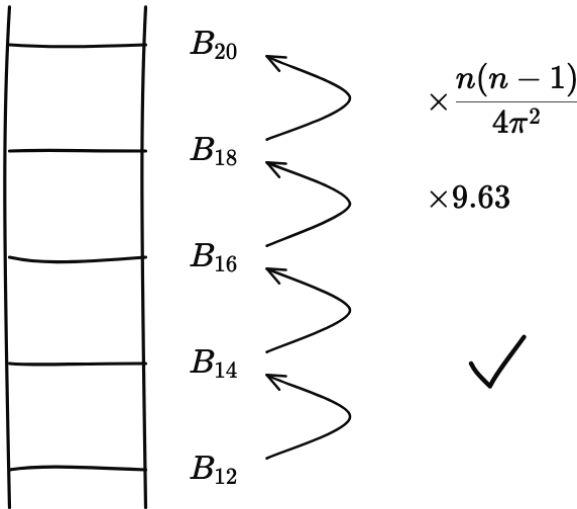
Measured from afar

His first instrument is a formula for the number of digits. It grows out of the bridge we met earlier, which ties each Bernoulli number to a factorial and a power of two pi. Taking logarithms, the tool for measuring the size of huge numbers, he turns that bridge into a short

recipe that needs only the index and two fixed constants. From it he can say how many digits a Bernoulli number has and what its first few digits are, and he even states how large the small error of the recipe is. For the tenth number in his table the recipe lands almost exactly on the true value of about five hundred and twenty-nine.

His second instrument compares neighbours. Each Bernoulli number turns out to be very nearly the one two places before it, multiplied by a simple factor built from the index and π . The approximation becomes excellent very quickly as the numbers grow. This gives a ladder: from one rung you can estimate the next well enough to know between which two whole numbers it lies. And since the leftover fraction is already known from the prime rule, knowing those two whole numbers is enough to write the exact value. Ramanujan climbs the first ten rungs this way, and every value agrees with his table.

WHOLE PART + KNOWN FRACTION



A ladder of neighbours

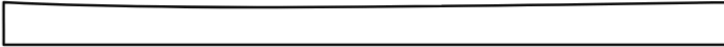
Then he shows off with a giant, the Bernoulli number with index four hundred and forty-four. Its leftover fraction comes from the prime rule, with a denominator of about ninety million. His digit recipe says its whole part has six hundred and thirty-one digits and tells him the first four; adding the size of the denominator, the full top of the fraction has six hundred and thirty-nine digits, and again he knows how it starts. Using his remainder rules, he even works out how the whole part and the top end. Modern software confirms these digits, although one step of his reasoning leans on his unreliable prime guess.

$$B_{444}$$

WHOLE: 631 DIGITS, 1751...



TOP: 639 DIGITS, 1588...



...657

BOTTOM: 90709710

ALSO: TAN, COSEC

SIZE FROM LOGS

$$\frac{B_n}{B_{n-2}} \approx \frac{n(n-1)}{4\pi^2}$$

A giant: B with index 444

Finally, he points out that everything he built on the cotangent can be rebuilt on the tangent or on the cosecant, two other functions from school trigonometry. That produces further identities connecting Bernoulli numbers spaced several places apart, now with powers of two mixed in, adding to his toolkit of shortcuts.

Remember

With logarithms, Ramanujan turns the bridge to factorials into a recipe for the number of digits and the first digits of any Bernoulli number. Each Bernoulli number is very nearly the one two places before it times a simple factor, so he can climb the table and fix exact values using the prime rule. He describes the number with index four hundred and forty-four, its length, first digits and last digits, without writing it out, and he extends his identities to the tangent and the cosecant.

Questions

1. What does Ramanujan's digit recipe let him do?
 - a. Say how many digits a Bernoulli number has and what its first digits are, from the index alone
 - b. Find the exact numerator of any Bernoulli number
 - c. Prove that the numerators are prime
 - d. Compute π
2. Why is knowing between which two whole numbers a Bernoulli number lies enough to get its exact value?
 - a. Because the leftover fraction is already known from the prime rule
 - b. Because Bernoulli numbers are whole numbers
 - c. Because the two numbers are always equal
 - d. Because the ladder is exact

3. What does he extend at the end of this stretch?
 - a. His identities, rebuilt on the tangent and the cosecant
 - b. His table to a hundred numbers
 - c. His proof of the prime guess
 - d. His digit recipe to odd indices

Answers

1. **a)** From it he can say how many digits a Bernoulli number has and what its first few digits are.
2. **a)** Since the leftover fraction is already known from the prime rule, knowing those two whole numbers is enough to write the exact value.
3. **a)** He points out that everything built on the cotangent can be rebuilt on the tangent or the cosecant, producing further identities.

Practice

1. Explain with your own analogy how a number can be described in detail without being written out.

Hint: Think of measuring something you cannot reach.

Answer: Like estimating a mountain's height from its shadow, you can learn a number's size, first digits and last digits from indirect evidence: a logarithm gives its length and opening digits, and divisibility rules give how it ends, without ever writing all its digits.

2. Why does the ladder of neighbours work better for large Bernoulli numbers than for small ones?

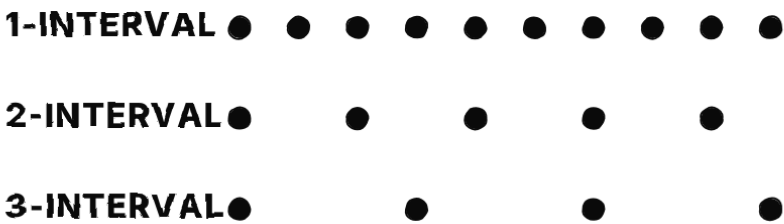
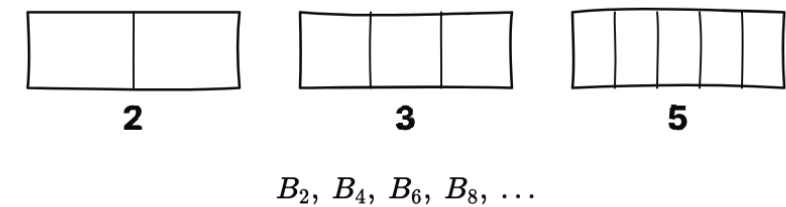
Hint: Think about the correction factors.

Answer: The neighbour factor is only approximately right, and the correction becomes extremely small as the numbers grow, so for large

indices the estimate is close enough to pick the correct pair of whole numbers every time.

Finer Sieves, and an Honest Verdict

If you have ever sorted coins with a set of sieves, you know the idea: a coarse sieve splits the pile in two, a finer one in three, and so on, until each tray holds only coins of one kind. Ramanujan's paper has been doing exactly this with Bernoulli numbers. In its last sections he names the method and shows how far it reaches.



$$\omega^p = 1 \text{ (or } \omega^p = i) \longrightarrow \cot$$

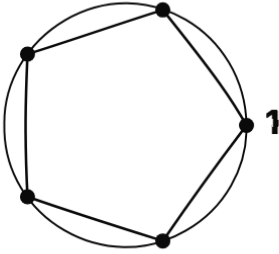
Sieves of every size

He calls his earlier results the one-interval, two-interval and three-interval formulas, after how far apart the Bernoulli numbers are that survive in each strand. The general recipe is short: to get a p-interval

formula, take the numbers whose p -th power is one, or whose p -th power is the imaginary unit when p is even, and mix them into the cotangent. Most terms then cancel, and each strand keeps only Bernoulli numbers spaced further and further apart, so each new one depends on fewer of the earlier ones.

To show the recipe working, he carries out the five-interval case. The five fifth roots of one sit at the corners of a regular pentagon, and the pentagon brings in the golden ratio, the number about one point six one eight famous from sunflowers, seashells and the Fibonacci numbers. His formulas use a sequence of whole numbers built from the golden ratio, close cousins of the Fibonacci numbers, and each formula keeps only every tenth Bernoulli number. He then sketches the four-interval case, built this time from the square root of two, and remarks that the same trick with cosines gives many other identities. For us the lesson is the method: one idea, cancelling terms with roots of one, generates a whole family of shortcuts, the way one good trick in a computer program can speed up many different calculations.

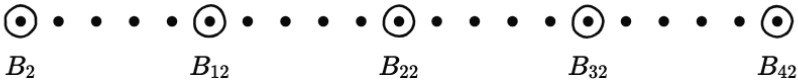
FIVE-INTERVAL



$$\varphi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

1, 3, 4, 7, 11, ...

four-interval: $\sqrt{2}$



ONE TRICK, MANY SHORTCUTS

The pentagon and phi

The book closes the paper with a note from the editors of Ramanujan's collected papers, written after his death. It is a respectful but frank verdict. The paper, they say, is interesting as a specimen of Ramanujan's earliest manner, but its principal results were already known and its proofs are incomplete. His rule for the leftover fractions is the theorem of Clausen and von Staudt, and his oddness pattern was known to Euler.

EDITORS

RESULTS KNOWN, PROOFS INCOMPLETE CLAUSEN-VON STAUDT, EULER

PROOF SKETCH ● ● ● ?

FOLLOWS ● ●

CONJECTURE

● ✓ DENOM RULE

● ✗ PRIME TOPS

●

SOME SUMS DIVERGE

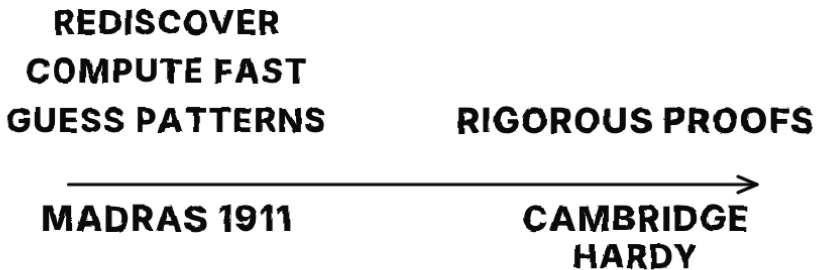
The editors' verdict

They also sort his eight arithmetic statements by their status, which is exactly the distinction this course has tried to keep. For three of them he indicated proofs, but those proofs rest on claims about whole-number coefficients that he never proved. Two more he presented as consequences of others. And three, including the claim that the tops of the fractions are prime and his rule for the denominators, he stated merely as conjectures; the rule for denominators is in fact true, while the prime claim is false. The editors add that some of his derivations use infinite sums that do not settle down to a value, so those arguments do not work as written.

Why read such a paper today? Because it is a rare look at a great mind before it was trained: a young man working alone in Madras,

rediscovering a famous theorem, inventing fast ways to compute, and guessing patterns from data with a sure instinct for which ones matter. Within a few years, working with Hardy in Cambridge, Ramanujan learned to support his insights with rigorous proofs, and his later work reshaped parts of number theory. This course has followed his first published paper from the cotangent to the editors' verdict, and it ends here.

WHY READ IT?



ROOTS OF ONE → FINER STRANDS
KNOWN RESULTS, PROOFS INCOMPLETE
DENOMINATORS ✓ PRIME TOPS ✕

From Madras to Cambridge

Remember

Ramanujan names his method interval formulas and shows the general recipe: roots of one mixed into the cotangent sort the Bernoulli numbers into ever finer strands, with the golden ratio appearing in the five-interval case. The editors' final note says the paper's main results were already known and its proofs are incomplete, and it separates his statements into proofs that rest on unproved claims, consequences, and plain conjectures. Some conjectures are true, like the rule for denominators, and one is false, the prime-top claim; the paper remains a vivid portrait of Ramanujan's earliest way of working.

Questions

1. What is the general recipe for a p -interval formula?
 - a. Mix into the cotangent the numbers whose p -th power is one, or the imaginary unit when p is even
 - b. Multiply the cotangent by p
 - c. Use the first p primes
 - d. Divide the table into p parts
2. What does the pentagon of fifth roots bring into the five-interval formulas?
 - a. The golden ratio
 - b. The number e
 - c. The square root of two

- d. The number π only
3. How do the editors sort Ramanujan's eight arithmetic statements?
- a. Three with indicated proofs resting on unproved claims, two consequences, and three conjectures
 - b. All eight fully proved
 - c. All eight false
 - d. Four proved and four false

Answers

1. a) To get a p-interval formula, take the numbers whose p-th power is one, or whose p-th power is the imaginary unit when p is even, and mix them into the cotangent.
2. a) The five fifth roots of one sit at the corners of a regular pentagon, and the pentagon brings in the golden ratio.
3. a) For three he indicated proofs resting on unproved claims, two he presented as consequences, and three he stated merely as conjectures.

Practice

1. In your own words, explain why finer sieves mean less work for each new Bernoulli number.

Hint: Think about how many earlier numbers sit in the same strand.

Answer: A finer sieve sorts the Bernoulli numbers into more strands, each with members spaced further apart. Each new number is computed only from earlier members of its own strand, so the finer the sieve, the fewer earlier numbers each step needs.

2. Explain the difference between a conjecture, a consequence and a proof that rests on an unproved claim, using the editors' note.

Hint: Think about what each one still needs.

Answer: A conjecture is a guess stated without proof; a consequence follows from another statement, so it is only as safe as that statement; a proof that rests on an unproved claim is complete except for that claim. The editors use these categories to show that none of Ramanujan's eight statements is fully proved in the paper, although several are true theorems.