

Collected Papers of Srinivasa Ramanujan



Srinivasa Ramanujan · 1927

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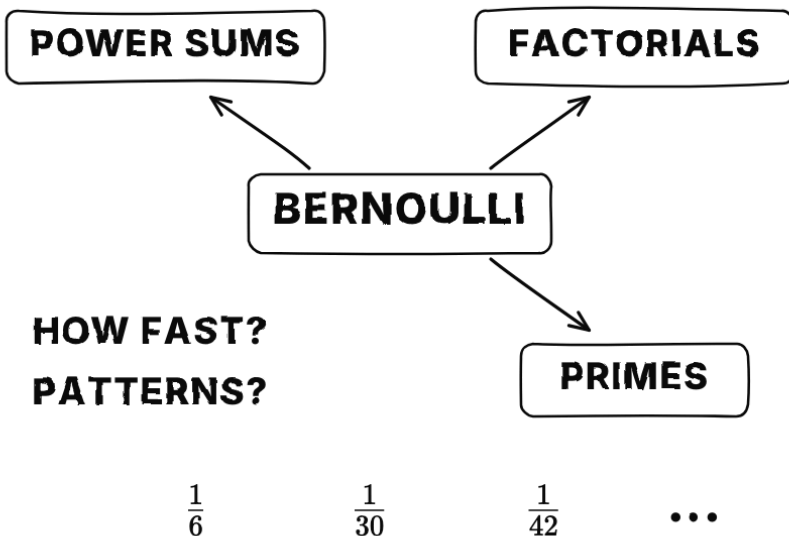
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Where the Bernoulli Numbers Live

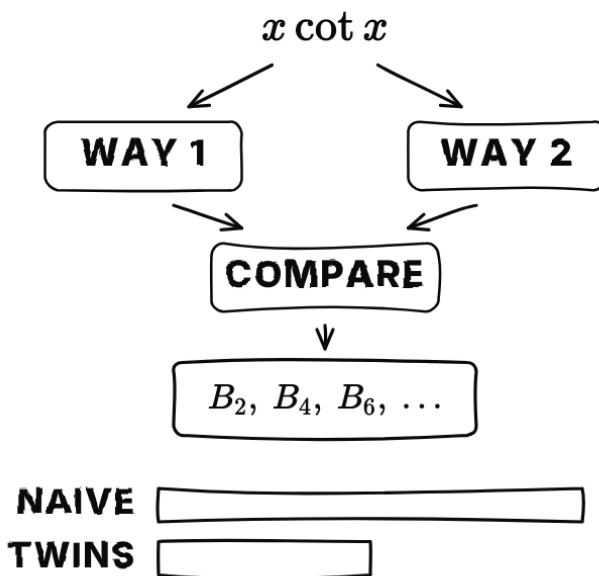
Some numbers keep turning up wherever you look in mathematics, and the Bernoulli numbers are among the most persistent. They appear when you add up the squares or cubes of whole numbers, when you estimate enormous factorials and when you study the prime numbers. The first ones are friendly fractions, one sixth, one thirtieth, one forty-second, but soon they grow wild, with numerators dozens of digits long. In 1911 a young clerk in Madras, Ramanujan, published his very first paper about them, and this course follows it.



Where they turn up

The paper asks two questions. How can these numbers be computed with as little effort as possible? And what hidden regularities do they obey, in their numerators, their denominators and their size? The opening sections deal with the first question, and the whole answer rests on one idea: a single familiar function contains all of them.

That function is the cotangent from school trigonometry, multiplied by the angle. Written out near zero as a long sum of powers, its successive terms carry the Bernoulli numbers one after another; Ramanujan takes them all as positive and calls the zeroth one minus one so that the first term fits. If you can describe this function in two different ways, the two descriptions must agree term by term, and every agreement is an equation that gives you the next Bernoulli number. That is the engine of the paper: write the same thing twice, compare, and read off the numbers.



Write it twice, compare

Done naively, the engine is slow. The first equations he gets need all of the earlier Bernoulli numbers to produce the next one, so the work grows steadily as you go. His improvement is to combine the cotangent with a twin version of itself, so that every second term cancels and each half of the family can be handled on its own. The new equations need only half as many earlier numbers, and the next part of the course pushes the same trick further.

Why does this matter beyond 1911? Choosing the combination that cancels most of the work is the same instinct behind the fast formulas inside a calculator or a computer algebra system, which compute these numbers today in a fraction of a second. And the values Ramanujan obtained this way are what led him to the patterns in the

later sections, some of which turned out to be true and famous, and some of which did not survive checking. Left out here are the actual equations, the three ways of writing the cotangent, the squared-series relation and the details of the twin function; they are in the first three sections of the paper.

Remember

The Bernoulli numbers are special fractions that appear throughout mathematics, and they all live inside the cotangent multiplied by the angle. Ramanujan's method is to describe that function in two ways and compare term by term, which hands out the numbers one by one. The plain comparison needs all the earlier numbers at each step; pairing the cotangent with its twin, so that half the terms cancel, cuts that work in half. This search for shortcuts, and the values it produced, set up the patterns the rest of the paper is about.

Questions

1. Which single idea do the opening sections of the paper rest on?
 - a. That the Bernoulli numbers are all prime
 - b. That one familiar function, the cotangent multiplied by the angle, contains all the Bernoulli numbers
 - c. That the Bernoulli numbers can be guessed from a table
 - d. That a computer is needed to find them

2. Why does describing the function in two ways produce the numbers?
 - a. Because the two descriptions must agree term by term, and each agreement is an equation
 - b. Because the two descriptions give two different numbers to average
 - c. Because one description is always simpler
 - d. Because the cotangent repeats itself
3. What does pairing the cotangent with its twin version achieve?
 - a. It makes every number an integer
 - b. Every second term cancels, so each half of the family can be handled with fewer earlier numbers
 - c. It doubles the work but improves accuracy
 - d. It proves all the patterns of the later sections

Answers

1. **b)** The whole answer rests on one idea: a single familiar function, the cotangent multiplied by the angle, contains all of them.
2. **a)** If you describe the function in two ways, the descriptions must agree term by term, and every agreement is an equation giving the next number.
3. **b)** Combining the cotangent with its twin cancels every second term, and the new equations need only half as many earlier numbers.

Practice

1. In your own words, explain why writing the same thing twice and comparing can produce new information.

Hint: Think of two descriptions of the same object that must agree in every detail.

Answer: If two expressions describe the same function, every piece of one must equal the matching piece of the other. Each such match is an equation, and when one side contains an unknown Bernoulli number while the other side contains only known quantities, the equation tells you the unknown. Repeating this hands out the numbers one by one.

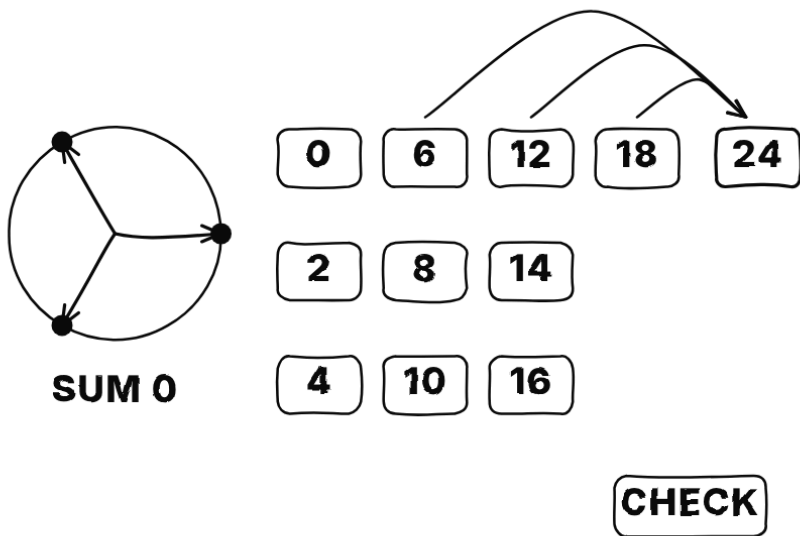
2. Why would Ramanujan care about needing fewer earlier numbers to find the next one?

Hint: Remember that he worked by hand and that the numbers grow quickly.

Answer: Each earlier number has to be carried into the calculation, and the numbers soon have long numerators, so every term costs real work by hand. Halving the number of terms needed halves that work and lowers the chance of an arithmetic mistake, which matters when you want values far along the sequence.

Faster Still, and the First Table

In the last part, Ramanujan paired the cotangent with a twin so that half of its terms cancelled. Now he asks the natural next question: if two partners cancel half the terms, could three partners cancel even more? The answer is yes, and it gives him a still faster recipe, plus a ready-made way to check his own arithmetic.



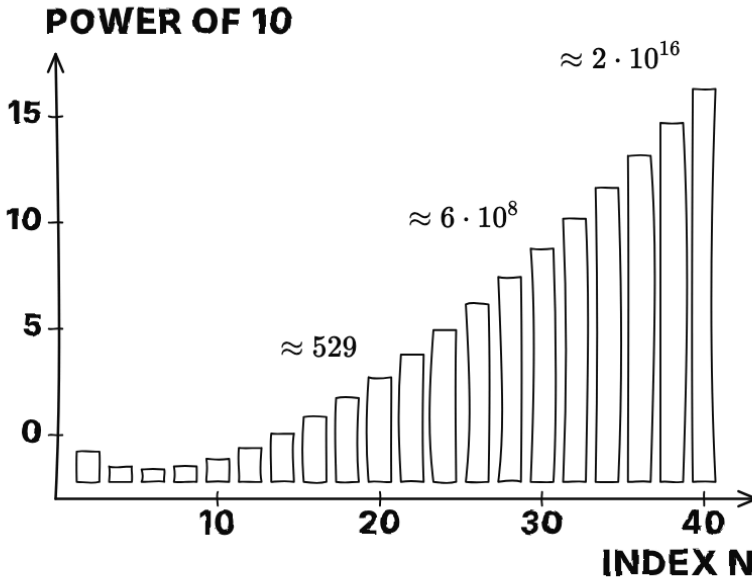
Three partners, a third of the work

The three partners come from the cube roots of one: one itself and two complex numbers that sit a third of a turn apart on a circle. Like three equal pulls on a ring at equal angles, these three numbers add

up to zero. Feeding all three into the cotangent and adding the results makes two terms out of every three cancel. What remains is a strand of Bernoulli numbers spaced six places apart, and by shifting the mixture he gets all three strands, each described by a simple ratio of two series.

The payoff is a recurrence, his twelfth result, in which each new Bernoulli number needs only a third of the earlier ones that the plain method of the first section required. To find the twenty-fourth, for instance, he needs only the eighteenth, the twelfth and the sixth. A second relation, built from products of pairs of these numbers, gives the same values by a different road, and Ramanujan says plainly that it is there for checking the calculations. Anyone who has added a long column twice, once upward and once downward, knows why that matters.

With this machinery he computes the first twenty Bernoulli numbers and lists them. They start gently, one sixth, one thirtieth, one forty-second, then one thirtieth again, but soon they climb steeply. The one with index twenty is about five hundred and twenty-nine, the one with index thirty is about six hundred million, and the one with index forty is about nineteen thousand million million. Their denominators stay modest while the numerators run to twenty or more digits, and the table ends with the remark that they grow without bound.



Small at first, then enormous

Why does this table matter? Because Ramanujan did not stop at computing. He looked at these values the way a naturalist looks at specimens, searching for regularities in which primes divide the numerators, which divide the denominators, and how the fractional parts behave. The next part of the course is about those patterns, and the table here is the evidence he drew them from. Today a computer algebra system produces the same table in an instant, but Ramanujan obtained it by hand, which is exactly why the speed of his recurrence mattered so much.

Left out here are the trigonometric identities with the cube roots, the three ratio formulas themselves, the exact form of the recurrence and

the check relation, and the intermediate equations he solved one by one; they are in sections four and five of the paper.

Remember

Three cube roots of one add up to zero, and mixing them into the cotangent cancels two terms in every three, leaving strands of Bernoulli numbers six places apart. This gives Ramanujan a faster recurrence, which needs only a third of the earlier numbers, together with a separate product relation for checking. With it he tabulates the first twenty Bernoulli numbers, which start as small fractions and then grow enormously. That table is the raw material for the patterns he states next.

Questions

1. Why does mixing the three cube roots of one into the cotangent cancel two terms in every three?
 - a. Because the three numbers add up to zero, like three equal pulls on a ring
 - b. Because each cube root is zero
 - c. Because the cotangent has only every third term
 - d. Because the cube roots are all equal
2. What is the second relation, built from products of pairs, used for?
 - a. To compute the numbers faster than the recurrence

- b. To check the arithmetic, since it reaches the same values by a different road
 - c. To prove that the numbers are prime
 - d. To find the denominators
3. Why does the table of twenty values matter for the rest of the paper?
- a. It is the evidence from which Ramanujan reads off his patterns
 - b. It proves all his later statements
 - c. It replaces the need for any formula
 - d. It shows that the numbers stay small

Answers

1. **a)** Like three equal pulls on a ring at equal angles, the three numbers add up to zero, so feeding all three into the cotangent cancels two terms in every three.
2. **b)** It gives the same values by a different road, and Ramanujan says it is there for checking the calculations.
3. **a)** Ramanujan looked at these values like a naturalist's specimens, and the table is the evidence for the patterns that follow.

Practice

1. Explain in your own words why a separate check relation is valuable when calculating by hand.

Hint: Think of adding a long column twice.

Answer: By hand, a single slip in a long calculation goes unnoticed and spoils every later value that depends on it. A second formula that reaches

the same numbers by a different road catches such slips: if both give the same fraction, an error is very unlikely.

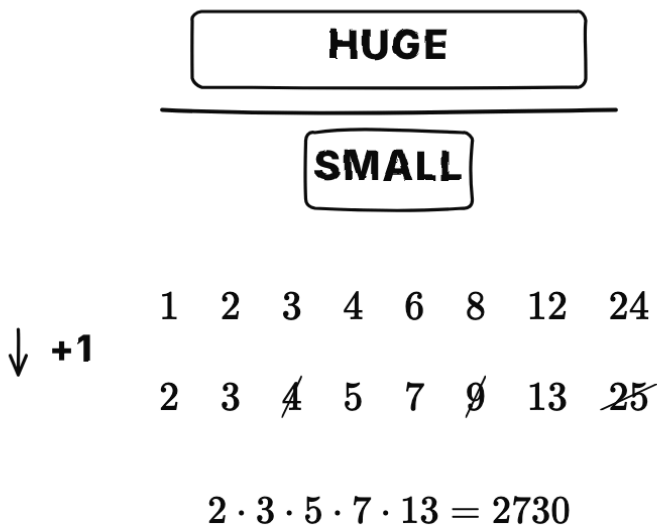
2. The table shows small denominators but rapidly growing numerators. What does that suggest a mathematician should look at next?

Hint: Think about what can be said about the top and the bottom of each fraction separately.

Answer: It suggests studying the two parts separately: which primes always appear in the small denominators, and what can be said about the enormous numerators. That is exactly where Ramanujan turns next, looking for rules connecting the index with the primes in the numerator and the denominator.

Reading Laws From the Specimens

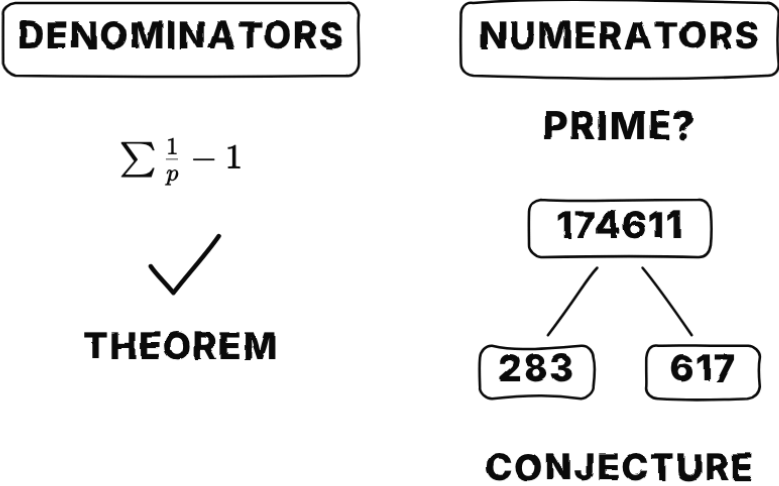
With his table of twenty Bernoulli numbers in hand, Ramanujan does what a naturalist does with a drawer of specimens: he looks for rules. The fractions have two very different halves. Their bottoms, the denominators, are small and tidy; their tops, the numerators, are huge and look random. So he asks two questions: what decides the denominator, and what can be said about the numerator?



The rule for the bottoms

For the denominators he finds a rule of startling simplicity. Take every number that divides the index, add one to each, keep only the

results that are prime, and multiply them together: that product is the denominator. For the index twenty-four this gives two thousand seven hundred and thirty, exactly the number in the table. A close cousin of the rule says that the part of each Bernoulli number after its whole part is made of one over each of those primes, added up, minus one. And from that it follows that when two Bernoulli numbers share the same denominator, their difference or their sum, depending on the pair, is a whole number. These denominator rules are true: they are a famous theorem found by Clausen and von Staudt decades earlier, which Ramanujan rediscovered on his own.



True rule, failed guess

For the numerators he is bolder. He observes that after dividing by the index, the numerator seems always to be a prime, and that a

prime dividing the index which does not appear in the denominator must appear in the numerator instead. He also notices that certain combinations of a Bernoulli number with powers of two always come out as odd whole numbers, and that some always leave a remainder of one when divided by thirty. The oddness pattern was already known to Euler. But the prime-numerator claim does not survive checking: for the index twenty, the numerator he calls prime is two hundred and eighty-three times six hundred and seventeen. The editors of his collected papers say plainly that several of these statements were conjectures, and that the proofs he sketches later depend on steps he never proved.

Why do patterns that came from twenty examples matter, true or not? Because this is how much of mathematics actually moves: someone stares at data, guesses rules, and others then prove or refute them. Ramanujan's guesses here mix rediscoveries of real theorems with a few hopeful extrapolations, a pattern that would mark his work for the rest of his short life, and the true ones are used to this day, for instance when computer algebra systems compute Bernoulli numbers exactly.

Left out here are the precise statements with their numbers, the worked examples for particular indices, the table of denominators, and the details of the divisibility and remainder rules; they are in sections six to nine of the paper.

Remember

From his table Ramanujan reads off rules for the tops and the bottoms of the Bernoulli numbers. The denominator is the product of the primes that are one more than a divisor of the index, and the leftover fraction is built from those same primes; these rules are a true theorem he rediscovered. His claims about the numerators are bolder: some, like the oddness patterns, are true and known, but the claim that the numerator is always prime fails, and the editors mark several statements as conjectures.

Questions

1. What rule does Ramanujan find for the denominators of the Bernoulli numbers?
 - a. Take the divisors of the index, add one, keep the primes, and multiply them
 - b. They are always powers of two
 - c. They equal the index plus one
 - d. They are random
2. Which of his patterns turned out to be a true, already known theorem?
 - a. That the numerator is always prime
 - b. The denominator rule, found by Clausen and von Staudt decades earlier

- c. That every Bernoulli number is a whole number
 - d. That the numbers shrink
3. What happened to the claim that the numerator is always a prime?
- a. It was proved by Ramanujan
 - b. It fails: for the index twenty the numerator is two hundred and eighty-three times six hundred and seventeen
 - c. It was confirmed by the editors
 - d. It holds for all indices above ten

Answers

1. **a)** Take every number that divides the index, add one to each, keep only the primes, and multiply them: that product is the denominator.
2. **b)** The denominator rules are true: they are a famous theorem of Clausen and von Staudt, which Ramanujan rediscovered.
3. **b)** The prime-numerator claim does not survive checking: for the index twenty the numerator he calls prime is two hundred and eighty-three times six hundred and seventeen.

Practice

1. In your own words, explain why guessing rules from a table of examples is useful even when some guesses are wrong.

Hint: Think about what happens to a guess after it is written down.

Answer: A stated rule can be tested by others: if it is true, someone proves it and it becomes a tool; if it is false, a counterexample shows exactly where it breaks, which also teaches something. Ramanujan's table produced both kinds: true theorems such as the denominator rule, and a prime-numerator claim that fails.

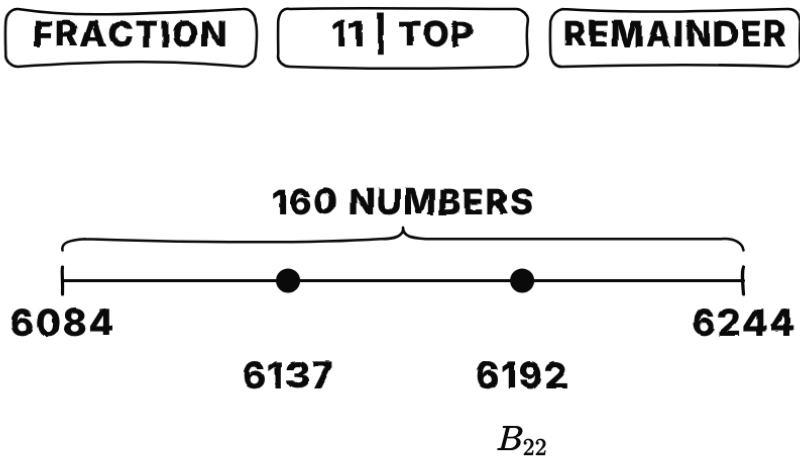
2. Why does it matter that the denominators follow a simple rule while the numerators do not?

Hint: Think about how one would compute a Bernoulli number exactly.

Answer: If the denominator is known in advance from the index alone, then computing a Bernoulli number exactly comes down to finding its numerator, and the fractional part is known too. That is why Ramanujan can later pin down an exact value from rough limits on its size.

From Rough Size to Exact Value, and Proofs Left Open

Here is a puzzle that captures this stretch of the paper. Someone tells you a fraction lies somewhere between six thousand and six thousand two hundred and fifty, tells you its bottom, and gives you a couple of facts about its top. Can you name it exactly? Ramanujan shows that for Bernoulli numbers the answer is often yes, and this is where all his earlier patterns pay off.



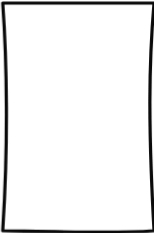
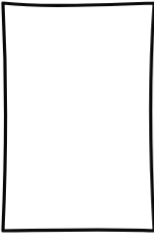
PRIME GUESS: FALSE

From a range to one value

His worked example is the Bernoulli number with index twenty-two. Knowing only rough limits on its size, he uses the rules from the previous sections: the leftover fraction after the whole part is fixed by the primes of the denominator, eleven must divide the top, and a remainder rule must hold. Out of a hundred and sixty whole numbers in the range, only two candidates survive, and he picks the right one, the value in his own table. There is a catch in how he picks it: he relies on his guess that the top, divided by eleven, is a prime, and that guess is false for this very number. So the idea of combining a rough size with exact arithmetic facts is sound and powerful, while one of the facts he leans on is not.

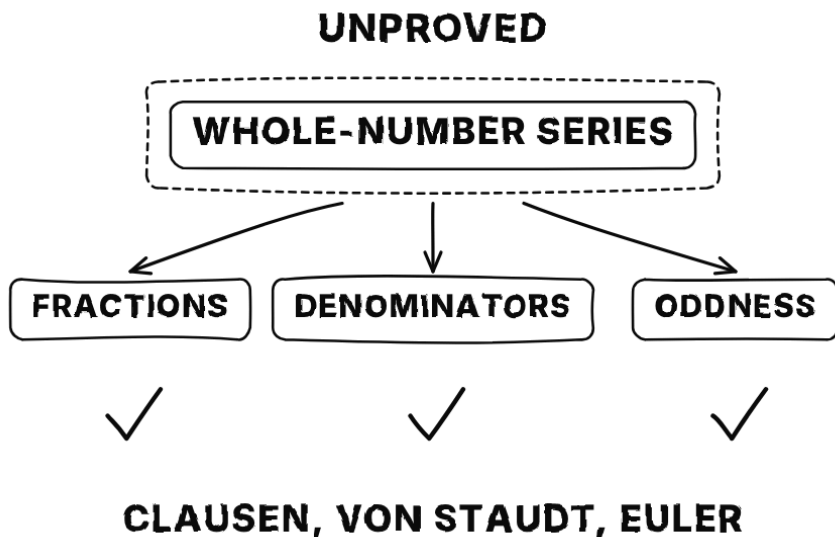
The next question is where the rough size comes from. Here Ramanujan brings in a classical bridge between Bernoulli numbers and the sum of one over every whole number raised to a power, the same sum that gives $\pi^2/6$ for the squares. It says that a Bernoulli number is essentially a factorial divided by a power of 2π , multiplied by a sum that is barely more than one. So the size of a Bernoulli number grows like a factorial, which explains the explosion in the table. He also turns the same fact into integral formulas, and plugs it into the classical expansion of the logarithm of a factorial, whose correction terms are themselves Bernoulli numbers. With enough decimal places of logarithms, this gives the whole part of any Bernoulli number, and the prime rule gives the fractional part. The practical difficulty, which he states honestly, is that the logarithm tables of his day did not have enough decimal places, a problem he solves a little later.

$$B_n \approx \frac{2n!}{(2\pi)^n} \\ \times \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right)$$

≈ 529.12  B_{20}	≈ 529.12  $\frac{2 \cdot 20!}{(2\pi)^{20}}$
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Why they grow like a factorial

Then he tries to prove his own patterns. His plan is to show that certain infinite sums, built from one over the squares of shifted whole numbers and from the primes above three, can be rewritten as series whose coefficients are all whole numbers. If that were true, the rule for the fractional parts and the rule about equal denominators would follow at once, and with two similar series, the oddness patterns too. It is an elegant strategy: reduce many arithmetic facts to one claim about whole-number coefficients.



One unproved claim

But the claim itself is never proved in the paper. The editors of the collected papers point out that the key series statements are left unproved and that some intermediate steps use series that do not converge, so the arguments are incomplete as they stand. The results he aimed at are nevertheless true and were known before him: the rule for the fractional parts is the theorem of Clausen and von Staudt, and the oddness pattern goes back to Euler. What remains impressive is that a young man working alone found the right statements and a plausible route to them.

Left out here are the integral formulas themselves, the expansion of the logarithm of the factorial with its remainder term, and the

detailed series identities of the attempted proof; they are in sections ten to twelve of the paper.

Remember

Rough limits on a Bernoulli number, combined with exact facts about its denominator and numerator, can pin down its exact value, though Ramanujan's example leans on a false prime guess. A classical formula ties Bernoulli numbers to factorials, powers of two π and sums of reciprocal powers, which explains their explosive growth and gives their whole part from precise logarithms. His attempt to prove the patterns reduces them to one claim about series with whole-number coefficients, a claim he never proves; the target results are true theorems found earlier by others.

Questions

1. How can a Bernoulli number be named exactly from only rough limits on its size?
 - a. By combining the limits with exact facts about its denominator and numerator
 - b. By guessing the first digit
 - c. By measuring it with a calculator
 - d. It cannot be done
2. Why do the Bernoulli numbers in the table explode in size?

- a. Because a Bernoulli number is essentially a factorial divided by a power of two π
 - b. Because their denominators shrink
 - c. Because of rounding errors
 - d. Because they double each time
3. What is the status of Ramanujan's attempted proof of his patterns?
- a. Complete and accepted
 - b. Incomplete: it rests on a claim about whole-number coefficients that he never proves
 - c. Disproved entirely
 - d. Copied from Euler

Answers

1. **a)** Knowing rough limits, he uses the rules from the previous sections, and out of a hundred and sixty candidates only two survive.
2. **a)** The classical bridge says a Bernoulli number is essentially a factorial divided by a power of two π , times a sum barely more than one, so it grows like a factorial.
3. **b)** His plan reduces the patterns to one claim about series with whole-number coefficients, and the claim itself is never proved in the paper.

Practice

1. In your own words, explain why combining a rough size with exact arithmetic facts is powerful.

Hint: Think of a lock whose combination you partly know.

Answer: The rough size limits the answer to a range of whole numbers, and each exact fact rules out most numbers in that range. Together they can leave a single possibility, just as knowing a few digits of a combination and the total of its digits can identify it, without ever computing the number directly.

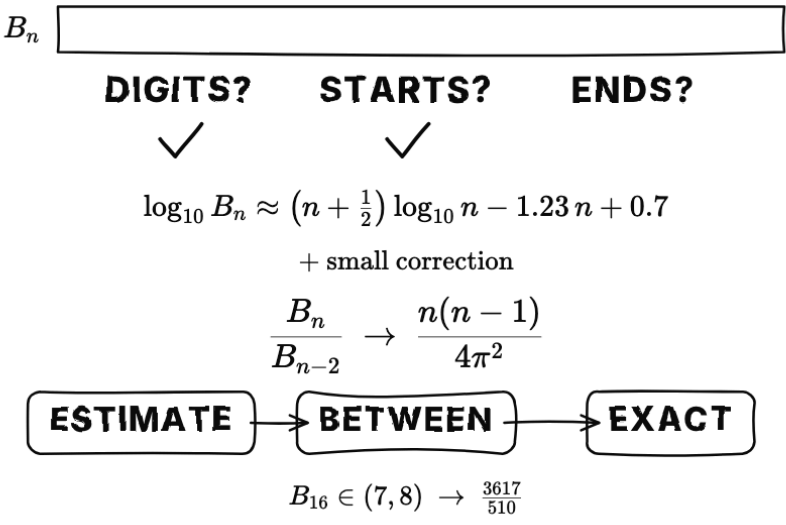
2. Why might Ramanujan's reduction of many patterns to one claim still be valuable, even though he did not prove the claim?

Hint: Think about what a later mathematician would need to do.

Answer: It shows exactly what would have to be proved: one statement about whole-number coefficients instead of many separate facts. Anyone who later proves that claim, or finds another proof of the results, knows where the real difficulty lies, and the results he aimed at turned out to be true theorems.

How Big, and What It Ends With

Suppose someone asks you about a number so large that writing it out would fill a page. You cannot compute it, but you can still say how many digits it has, what it starts with, and even what it ends with. That is the trick of this stretch of the paper, applied to Bernoulli numbers far beyond Ramanujan's table.



Size, then a ratio

First, the size. From the bridge between Bernoulli numbers and factorials, Ramanujan derives a simple approximate formula for the number of digits. It needs only ordinary logarithms of the index, a

couple of fixed constants, and a small correction that he also states. So for any index, however large, he can say how many digits the whole part has and what its first few digits are, without computing the number itself.

Second, a ratio. Comparing each Bernoulli number with the one two places before it, he shows that their ratio settles very quickly onto a simple expression built from the index and π . That gives a practical way to walk up the table: estimate the next number from the previous one, find which two whole numbers it lies between, and then add the leftover fraction from the prime rule to get the exact value. He shows this step by step for the first ten numbers, each one obtained from the one before.

Third, the showpiece: the Bernoulli number with index four hundred and forty-four. In 1911 writing it out by hand was out of the question, yet Ramanujan describes it in detail. Its leftover fraction comes from the prime rule; its whole part has six hundred and thirty-one digits starting with one, seven, five, one; its full numerator has six hundred and thirty-nine digits starting with one, five, eight, eight. Using his remainder rules he even works out how the whole part and the numerator end. It is like describing a person you have never met from their footprints: height, weight and gait, all from indirect evidence. Part of this description leans on his unproved prime-numerator guess, but the size estimates and the fractional part rest on solid ground.

$$B_{444}$$

TOO BIG TO WRITE OUT

WHOLE: 631 DIGITS, 1751...



TOP: 639 DIGITS, 1588...

...4 ?



...57 ?

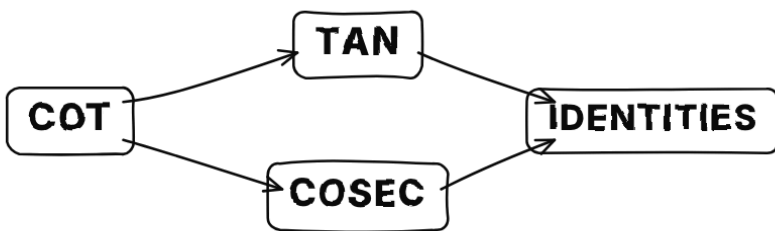
FRACTION: PRIME RULE



LIKE READING FOOTPRINTS

The giant B_{444}

Finally, he notes that everything he did with the cotangent can be repeated with the tangent or the cosecant, giving still more identities among the Bernoulli numbers, with other spacings and other powers of two.



DETAILS: SECTIONS 13–16

DIGITS FROM LOGS

$$\frac{B_n}{B_{n-2}} \approx \frac{n(n-1)}{4\pi^2}$$

B_{444} : length, start, end

Beyond the cotangent

Left out here are the digit formula with its constants, the ratio formula with its product over primes, the step-by-step table of the first ten numbers, and the explicit tangent and cosecant identities; they are in sections thirteen to sixteen of the paper.

Remember

Ramanujan gives a simple formula for how many digits a Bernoulli number has and how it begins, valid for any index. The ratio of each Bernoulli number to the one two places earlier quickly settles onto a simple expression with π , which lets him walk up the table and fix each exact value with the prime rule. He then describes the Bernoulli number with index four hundred and forty-four, its number of digits, its first digits and how it ends, without ever writing it out, and he extends his identities to the tangent and the cosecant.

Questions

1. What can Ramanujan's digit formula tell you about a Bernoulli number without computing it?
 - a. How many digits its whole part has and what its first few digits are
 - b. Its exact numerator
 - c. Whether it is prime
 - d. Its last digit only
2. How does the ratio rule help to walk up the table?
 - a. Estimate the next number from the previous one, find the two whole numbers it lies between, then add the leftover fraction from the prime rule
 - b. Multiply each number by two

- c. Add the previous two numbers
 - d. Divide by pi each time
3. Which part of the description of the four hundred and forty-fourth Bernoulli number is on shaky ground?
- a. The part that leans on his unproved prime-numerator guess
 - b. The number of digits
 - c. The leftover fraction
 - d. The first digits

Answers

1. **a)** For any index, however large, he can say how many digits the whole part has and what its first few digits are, without computing the number itself.
2. **a)** Estimate the next number from the previous one, find which two whole numbers it lies between, and then add the leftover fraction from the prime rule to get the exact value.
3. **a)** Part of this description leans on his unproved prime-numerator guess, while the size estimates and the fractional part rest on solid ground.

Practice

1. In your own words, explain how one can know the first digits and the number of digits of a number too large to write out.

Hint: Think about what a logarithm tells you.

Answer: The logarithm of a number tells you its size: its whole part gives the number of digits, and its fractional part gives the first digits. If a formula estimates the logarithm accurately, you learn both without ever writing the number.

2. Why is it useful that the ratio of successive Bernoulli numbers settles quickly onto a simple expression?

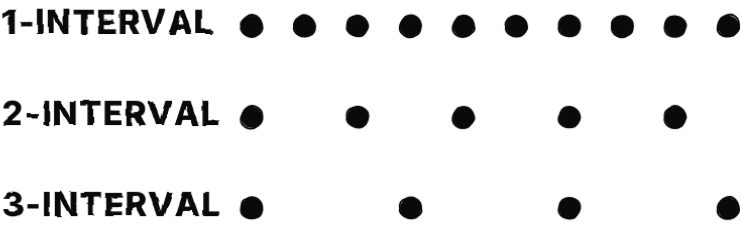
Hint: Think about how you would find the next number in the table.

Answer: It lets you estimate each new Bernoulli number from the previous one closely enough to know between which two whole numbers it lies. Since the leftover fraction is known from the prime rule, that is enough to write down the exact value.

The General Pattern, and What the Editors Said

By now a pattern runs through the whole paper. With one cotangent, Ramanujan got equations that use all the earlier Bernoulli numbers; pairing it with a twin cancelled half the terms; three cube roots of one cancelled two terms in every three. In the last two sections he names this pattern and pushes it as far as it goes.

$$B_2, B_4, B_6, B_8, \dots$$



$$p\text{-th roots of } 1 \rightarrow p\text{-interval}$$



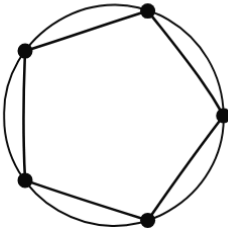
One pattern, finer sieves

He calls these results the one-interval, two-interval and three-interval formulas, after how far apart the surviving Bernoulli numbers sit. And

he states the general recipe: for any number p , take the p -th roots of one, or of the imaginary unit when p is even, mix them into the cotangent, and you get p -interval formulas, each keeping only every p -th member of the family in its own strand. It is like a set of sieves, each finer than the last, sorting the same pile of numbers into more and more separate trays.

To show it working, he carries out the five-interval case. The fifth roots of one bring in the golden ratio, the number about one point six one eight that appears in sunflowers, in the Fibonacci numbers and in classical architecture. The formulas are written with a sequence of whole numbers built from the golden ratio, close cousins of the Fibonacci numbers, and each formula isolates the Bernoulli numbers spaced ten indices apart. He then sketches the four-interval case too, this time built from the square root of two, and remarks that similar identities with cosines give many more. For a reader today the lesson is the method rather than the formulas: a single idea, cancelling terms with roots of one, generates a whole family of shortcuts, much as one clever trick in a computer program can speed up many calculations at once.

FIVE-INTERVAL



$$\varphi \approx 1.618$$

$$1, 3, 4, 7, 11, \dots$$



$$\sqrt{2} \rightarrow \text{four-interval}$$

ONE TRICK, MANY SHORTCUTS

Pentagon and golden ratio

The paper ends there, but the book adds a final word from the editors of Ramanujan's collected papers, including Hardy, who knew his work best. Their verdict is respectful and frank. The paper is interesting as a specimen of Ramanujan's earliest manner, but its principal results were already known, and its proofs are incomplete. The rule for the leftover fractions is the theorem of Clausen and von Staudt, and the oddness pattern was known to Euler. Of his eight arithmetic statements, he indicated proofs for only three, and those proofs depend on claims about whole-number coefficients that he never proved; two more are corollaries, and three, including the prime-numerator claim and the denominator rule, were stated merely as conjectures. The editors also note that some of his derivations use

series that do not converge, so the arguments are invalid as they stand.

EDITORS, WITH HARDY
RESULTS KNOWN, PROOFS INCOMPLETE
CLAUSEN–VON STAUDT, EULER

PROOF?	COROLLARY	CONJECTURE
●	●	●
●	●	●
●		●

× **SOME SERIES DIVERGE**

The editors' verdict

Why is this still worth reading? Because it shows a mathematical mind at the very start: self-taught, without access to the literature, rediscovering a famous theorem, finding fast computational shortcuts, and guessing patterns from data with a sure instinct for which ones matter. A few years later, working with Hardy, Ramanujan would add the rigour that this first paper lacks, and his later work changed number theory. This course has followed his first published paper from the cotangent to the editors' verdict; it ends here.

WHY READ IT?

REDISCOVER

SHORTCUTS

GUESSES

RIGOUR

SELF-TAUGHT
1911

WITH HARDY

ROOTS OF ONE → FINER STRANDS

KNOWN RESULTS, PROOFS INCOMPLETE

FAST METHODS, BOLD GUESSES

Before the rigour

Remember

Ramanujan generalises his trick: the p -th roots of one, mixed into the cotangent, give p -interval formulas that sort the Bernoulli numbers into ever finer strands, shown in full for the five-interval case built on the golden ratio. The editors' final note says the paper's main results were already known, its proofs are incomplete, and several of its arithmetic statements are only conjectures, one of them false. What remains is a vivid picture of Ramanujan's earliest manner: fast methods, rediscovered theorems and bold guesses, before the rigour he later learned.

Questions

1. What is Ramanujan's general recipe for p-interval formulas?
 - a. Take the p-th roots of one, or of the imaginary unit when p is even, and mix them into the cotangent
 - b. Multiply the cotangent by p
 - c. Use p different primes
 - d. Divide the table into p equal parts
2. What enters the five-interval formulas?
 - a. The golden ratio, through whole numbers built from it
 - b. The square root of two only
 - c. The number e
 - d. The prime number five alone
3. What is the editors' verdict on the paper?
 - a. Interesting as a specimen of Ramanujan's earliest manner, but its main results were known and its proofs are incomplete
 - b. A complete proof of new results
 - c. Entirely false
 - d. Unreadable

Answers

1. a) For any number p, take the p-th roots of one, or of the imaginary unit when p is even, mix them into the cotangent, and you get p-interval formulas.

2. a) The fifth roots of one bring in the golden ratio, and the formulas are written with a sequence of whole numbers built from it.

3. a) The paper is interesting as a specimen of Ramanujan's earliest manner, but its principal results were already known, and its proofs are incomplete.

Practice

1. In your own words, explain the sieve picture of the interval formulas.

Hint: Think of what each finer sieve does to the pile of Bernoulli numbers.

Answer: Each interval formula keeps only Bernoulli numbers spaced a fixed distance apart, so the family is sorted into separate strands. A finer sieve, from more roots of one, makes the strands thinner, so each new number in a strand depends on fewer earlier ones.

2. Why might a paper with incomplete proofs and some false conjectures still be worth studying?

Hint: Think about what it shows about how mathematics is discovered.

Answer: It shows discovery in action: fast methods, a famous theorem rediscovered independently, and bold guesses from data, some right and some wrong. Seeing which guesses survived, and why the proofs fall short, teaches how to judge mathematical arguments, and it shows the starting point of a mathematician whose later work, made rigorous with Hardy, changed number theory.